Algebraicity of Artin–Hasse–Weil L-series over global function fields

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Abstract

We prove an analogue of Deligne's period conjecture for the special value of the L-function of an abelian variety over a global function field twisted by an Artin representation. We illustrate this in action for an example of an elliptic curve twisted by a Dirichlet character.

Deligne's period conjecture is an abstract statement on the special value of the L-function associated to a pure motive with a critical Hodge structure [Del79, Definition 1.3]. Specifically, he conjectures that the L-value is equal to the determinant of a certain period map between its Betti and de Rham realisations, up to non-zero multiples in a number field [Del79, Conjecture 2.8]. This is known for Artin L-functions over $\mathbb{Q}$ [Del79, Proposition 6.7], and has ramifications for the Birch–Swinnerton-Dyer conjecture for abelian varieties over $\mathbb{Q}$ [Del79, Section 4], with numerical evidence for L-functions associated to Jacobians of smooth projective curves over $\mathbb{Q}$ [ECW24, Conjecture 1.1].

In the context of the L-function $L(A, \tau, s)$ of an abelian variety A over a number field K twisted by an Artin representation τ over K, which appear in equivariant refinements of the Birch–Swinnerton-Dyer conjecture [BC24, Conjecture 3.3], Deligne's period conjecture translates to a statement on the algebraicity and Galois equivariance of $L(A, \tau, 1)$ normalised by periods [Eva21, Proposition 4.3.8]. This remains largely open in general, but the case of an elliptic curve over $\mathbb{Q}$ twisted by Artin representations that factor through a false Tate curve extension, such as the trivial representation and primitive Dirichlet characters, is a consequence of the modularity theorem [BD07, Theorem 4.2].

When A is an abelian variety over a global function field K, the works of Grothendieck [Gro95, Theorem 5.1] and Deligne [Del73, Theorem 9.3] show that $L(A, \tau, s)$ is already a rational function satisfying a globally compatible functional equation, so that the aforementioned normalisations by periods are unnecessary. This paper presents a short proof of the analogue of Deligne's period conjecture in this context, which is stated in Theorem 2. To this end, some notational conventions for the formalism of ℓ-adic representations over local fields and global function fields will first be established. Throughout, ℓ will be a fixed prime of $\mathbb{Q}$, and V_ℓ will be a finite-dimensional vector space over a finite extension of $\mathbb{Q}_\ell$, whose choice will not be essential.

Notation 1. Let F be a non-archimedean local field with residue characteristic p. Let I_F denote the inertia subgroup of its Weil group W_F, and let Fr_F denote the inverse of any choice of Frobenius element in W_F. For $\ell \neq p$, an *ℓ-adic representation over F* is a continuous homomorphism $\rho : W_F \to \mathrm{GL}(V_\ell)$. Its *Euler factor* is the inverse characteristic polynomial

$$\mathcal{L}_F(\rho, T) \coloneqq \det(1 - T \cdot \mathrm{Fr}_F \mid \rho^{I_F}),$$

where ρ^{I_F} is the subrepresentation of ρ invariant under I_F.

Now let K be the global function field of a smooth proper geometrically irreducible curve C of genus g_C over a finite field $\mathbb{F}_q$ with absolute Galois group G_K. For each place v of K, let K_v denote its completion, and let $\deg v$ denote its residue class degree. For $\ell \nmid q$, an *ℓ-adic representation over K* is a continuous homomorphism $\rho : G_K \to \mathrm{GL}(V_\ell)$. Its *formal L-series* is the infinite product

$$\mathcal{L}(\rho, T) \coloneqq \prod_v \frac{1}{\mathcal{L}_{K_v}(\rho, T^{\deg v})},$$

which is a priori only a formal product. Its *L-series* $L(\rho, s)$ is simply $\mathcal{L}(\rho, q^{-s})$, and let $L^{(n)}(\rho, s)$ denote the n-th derivative of $L(\rho, s)$ for all $n \in \mathbb{N}$. Let $\mathfrak{f}_\rho$ denote the Artin conductor of ρ, and let $\deg \mathfrak{f}_\rho$ denote its degree as a Weil divisor on C. Finally, let G_K^g denote the geometric Galois group, namely the kernel of the natural restriction from G_K to the absolute Galois group of $\mathbb{F}_q$, and let $\rho^{G_K^g}$ denote the subrepresentation of ρ invariant under G_K^g.

The key example of an ℓ-adic representation over K will be the first ℓ-adic cohomology group $\rho_A \coloneqq H^1_{\text{ét}}(A, \mathbb{Q}_\ell)$ of an abelian variety A over K, which is independent of ℓ [GR72, Theorem 4.3], so ℓ is suppressed from notation. Another example is an *Artin representation*, namely a continuous homomorphism $\tau : G_K \to \mathrm{GL}(V)$, where V is a finite-dimensional vector space over a number field equipped with the discrete topology, viewed as an ℓ-adic representation over K by some embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$. The relevant notions over F are defined analogously. Finally, let $L^{(n)}(A, \tau, s)$ denote $L^{(n)}(\rho_A \otimes \tau, s)$ for all $n \in \mathbb{N}$.

For an Artin representation τ, let $\mathbb{Q}(\tau)$ denote the number field generated by the values of $\mathrm{tr}(\tau)$, and let τ^σ denote the representation with character $\sigma \circ \mathrm{tr}(\tau)$ for any $\sigma \in G_\mathbb{Q}$. If $(v_i)_i$ is a basis of τ over $\mathbb{Q}$, and (a_{ij}) is the matrix of $g \in G_K$ with respect to this basis, then $(v_i^\sigma)_i$ is a basis of τ^σ over $\mathbb{Q}$, and the matrix of g with respect to this basis is (a_{ij}^σ) [Eva21, Section 2.1.4].

The main result of this paper is as follows.

Theorem 2. Let A be an abelian variety over a global function field K, and let τ be an Artin representation over K. Then $L^{(n)}(A, \tau, 1) \in \mathbb{Q}(\tau)$ and $L^{(n)}(A, \tau, 1)^\sigma = L^{(n)}(A, \tau^\sigma, 1)$ for any $\sigma \in G_\mathbb{Q}$, for all $n \in \mathbb{N}$.

It turns out that the same argument applies to Artin L-series.

Theorem 3. Let τ be an Artin representation over K. Then $L^{(n)}(\tau, 1) \in \mathbb{Q}(\tau)$ and $L^{(n)}(\tau, 1)^\sigma = L^{(n)}(\tau^\sigma, 1)$ for any $\sigma \in G_\mathbb{Q}$, for all $n \in \mathbb{N}$.

In what follows, a stronger result on the algebraicity of formal L-series will be proven, which clearly implies the same for all derivatives of L-series, by replacing T with q^{-s}. To this end, for any field $\mathfrak{F}$ with automorphism group G, define an action of G on the ring of formal power series $\mathfrak{F}[[T]]$ by

$$\left(\sum_{n=0}^{\infty} a_n T^n\right)^g \coloneqq \sum_{n=0}^{\infty} a_n^g T^n, \qquad g \in G.$$

Evaluating such a power series at an element $f \in \mathfrak{F}$ does not give $\sum_{n=0}^{\infty} a_n^g f^n$ in general, due to potential convergence issues, but the following shows that it does whenever the power series happens to be a rational function.

Lemma 4. Let $\mathfrak{F}$ be a field, and let $P(T) \in \mathfrak{F}[[T]]$ be a power series such that $P(T) = R(T)\,/Q(T)$ for some power series $Q(T) \in \mathfrak{F}[[T]]$ and some polynomial $R(T) \in \mathfrak{F}[T]$. Then $P(T)^\sigma = R(T)^\sigma\,/Q(T)^\sigma$.

Proof. Since $R(T)$ is a polynomial, it suffices to show that $(P(T)\,Q(T))^\sigma = P(T)^\sigma\,Q(T)^\sigma$. Let P_n and Q_n denote the coefficients of the power series $P(T)$ and $Q(T)$ respectively for all $n \in \mathbb{N}$, so that the equality becomes

$$\sum_{n=0}^{\infty}\left(\sum_{i+j=n} P_i Q_j\right)^\sigma T^n = \sum_{n=0}^{\infty} P_n^\sigma T^n \cdot \sum_{n=0}^{\infty} Q_n^\sigma T^n.$$

This is clear since $\sum_{i+j=n} P_i Q_j$ is a finite sum. □

This is a property inherited by the formal L-series of a general ℓ-adic representation over a global function field. Furthermore, if it has no geometric invariants, its formal L-series is in fact a polynomial.

Proposition 5. Let ρ be an ℓ-adic representation over a global function field $K = \mathbb{F}_q(C)$ that is unramified almost everywhere. Then $\mathcal{L}(\rho, T) \in \overline{\mathbb{Q}_\ell}(T)$. Furthermore, if $\rho^{G_K^g} = 0$, then $\mathcal{L}(\rho, T) \in \overline{\mathbb{Q}_\ell}[T]$ of degree

$$\deg \mathcal{L}(\rho, T) = (2g_C - 2)\dim\rho + \deg \mathfrak{f}_\rho.$$

Proof. This follows the sketch of an argument in Ulmer's notes [Ulm11, Lecture 4, Theorem 1.4.1], but it is repeated here with references to Milne's book. There is an equivalence of categories between continuous ℓ-adic representations over K that are unramified on an open set U of C and ℓ-adic sheaves that are lisse on U [Mil80, Chapter V, Section 1]. Let $\iota : U \hookrightarrow C$ be any open set at which ρ is unramified, and let $\mathcal{F}_\rho$ be its associated ℓ-adic sheaf that is lisse on U, whose direct image along ι induces étale cohomology groups $H^i \coloneqq H^i_{\text{ét}}(\overline{C}, \iota_* \mathcal{F}_\rho)$ of the base change $\overline{C}$ of C to $\overline{\mathbb{F}_q}$. Now the Grothendieck–Lefschetz trace formula for ℓ-adic sheaves [Mil80, Chapter VI, Theorem 13.4] says that for all $n \in \mathbb{N}$,

$$\sum_{v \in C(\mathbb{F}_{q^n})} \operatorname{tr}(\mathrm{Fr}_{K_v}^n \mid \rho^{I_{K_v}}) = \sum_{i=0}^{2} (-1)^i \cdot \operatorname{tr}(\mathrm{Fr}_q^n \mid H^i),$$

where Fr_q is the Frobenius in $\mathbb{F}_q$. Dividing both sides by n and exponentiating their generating functions, this equality rearranges to

$$\prod_v \exp \sum_{m=1}^{\infty} \mathrm{tr}(\mathrm{Fr}_{K_v}^m \mid \rho^{I_{K_v}}) \frac{T^{m \deg v}}{m} = \prod_{i=0}^{2} \exp \left(\sum_{n=1}^{\infty} \mathrm{tr}(\mathrm{Fr}_q^n \mid H^i) \frac{T^n}{n} \right)^{(-1)^i} .$$

An identity in linear algebra [Mil80, Chapter V, Lemma 2.7] shows that

$$\exp \sum_{m=1}^{\infty} \mathrm{tr}(\mathrm{Fr}_{K_v}^m \mid \rho^{I_{K_v}}) \frac{T^{m \deg v}}{m} = \frac{1}{\det(1 - T^{\deg v} \cdot \mathrm{Fr}_{K_v} \mid \rho^{I_{K_v}})},$$

for each place v of K, and that

$$\exp \sum_{n=1}^{\infty} \mathrm{tr}(\mathrm{Fr}_q^n \mid H^i) \frac{T^n}{n} = \frac{1}{\det(1 - T \cdot \mathrm{Fr}_q \mid H^i)}.$$

for $i = 0, 1, 2$. Thus the left hand side becomes $\mathcal{L}(\rho, T)$, while the right hand side expresses it as an alternating product of polynomials $\det(1 - T \cdot \mathrm{Fr}_q \mid H^i)$, which proves the first statement. For the second statement, note that

$$H^0 = H^0_{\text{ét}}(\overline{U}, \mathcal{F}_\rho) \cong \rho^{G_K^g},$$

by definition, and Poincaré duality [Mil80, Chapter V, Proposition 2.2(c)] gives

$$H^2 \cong H^0_{\text{ét}}(\overline{C}, \iota_* \mathcal{F}_\rho^\vee(1))^\vee = H^0_{\text{ét}}(\overline{U}, \mathcal{F}_\rho^\vee(1))^\vee \cong \left(\rho^\vee(1)^{G_K^g}\right)^\vee = 0,$$

so that $\mathcal{L}(\rho, T) \in \overline{\mathbb{Q}_\ell}[T]$. Its degree is precisely given by the Grothendieck–Ogg–Shafarevich formula [Mil80, Chapter V, Theorem 2.12]. □

In particular, $\mathcal{L}(\rho, T)$ is well-defined at $T = q^{-1}$ and respects the action of automorphisms of $\overline{\mathbb{Q}_\ell}$ whenever its denominator does not vanish.

Remark 6. When ρ is the ℓ-adic representation associated to an elliptic curve, Shioda gave an alternative description of $\mathcal{L}(\rho, T)$ in terms of its associated elliptic surface $\mathcal{E}$ [Shi92, Theorem 4]. When $\mathcal{E}$ has at least one singular fibre, he showed that $\mathcal{L}(\rho, T)$ is in fact a polynomial, given by

$$\mathcal{L}(\rho, T) = \det(1 - T \cdot \mathrm{Fr}_q \mid W),$$

where W is a subspace of the second ℓ-adic cohomology group $H^2_{\text{ét}}(\mathcal{E}, \mathbb{Q}_\ell(1))$ of $\mathcal{E}$, given as the orthogonal complement of the trivial sublattice of the Neron–Severi group $\mathrm{NS}(\mathcal{E})$ of $\mathcal{E}$ under the cycle class map $\mathrm{NS}(\mathcal{E}) \to H^2_{\text{ét}}(\mathcal{E}, \mathbb{Q}_\ell(1))$. His description has the added benefit that the degree and functional equation of the polynomial $\mathcal{L}(\rho, T)$ can be understood directly from the geometry of $\mathcal{E}$.

The analogue of algebraicity and Galois equivariance can first be proven at the level of Euler factors of local ℓ-adic representations.

Proposition 7. Let ρ be an ℓ-adic representation over a non-archimedean local field F, such that $\mathcal{L}_F(\rho, T)$ has coefficients in $\mathbb{Q}$, and let τ be an Artin representation over F. Then $\mathcal{L}_F(\rho \otimes \tau, T) \in \mathbb{Q}(\tau)[T]$ and $\mathcal{L}_F(\rho \otimes \tau, T)^\sigma = \mathcal{L}_F(\rho \otimes \tau^\sigma, T)$ for any $\sigma \in G_\mathbb{Q}$.

Proof. This is similar to the argument by Bouganis–Dokchitser, but they only proved it for Artin twists of elliptic curves over number fields [BD07, Lemma 4.4], so it is repeated here for reference. The first statement follows from the second statement since $\tau^\sigma = \tau$ for any $\sigma \in G_{\mathbb{Q}(\tau)}$, so it suffices to prove the latter. Since $\mathcal{L}_F(\rho \otimes \tau, T)$ has coefficients in $\mathbb{Q}(\tau)$, it suffices to prove it for $\sigma \in \mathrm{Gal}(\mathfrak{L}/\mathfrak{K})$, where $\mathfrak{L}$ is the extension of $\mathbb{Q}(\tau)$ that realises τ and $\mathfrak{K}$ is its subfield fixed by σ. There is an equivalence of categories between ℓ-adic representations over F and complex Weil–Deligne representations of W_F [Del73, Section 8], so that ℓ can be replaced with some prime ℓ' that splits in $\mathfrak{K}$ and remains inert in $\mathfrak{L}$, which exists by Chebotarev's density theorem. This gives an isomorphism $\phi : \mathrm{Gal}(\mathfrak{L}/\mathfrak{K}) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}_{\ell'}(\alpha)/\mathbb{Q}_{\ell'})$, where α is the image in $\overline{\mathbb{Q}_{\ell'}}$ of the primitive element of $\mathfrak{K}$ that generates $\mathfrak{L}$. Now let $(v_i)_i$ be a basis of $(\rho \otimes \tau)^{I_F}$ over $\mathbb{Q}_\ell$, and let (a_{ij}) be the matrix of Fr_F with respect to this basis. Then $(v_i^{\phi(\sigma)})_i$ is a basis of $(\rho \otimes \tau^\sigma)^{I_F}$ over $\mathbb{Q}_\ell$, and the matrix of Fr_F with respect to this basis is $(a_{ij}^{\phi(\sigma)})$, so that its inverse characteristic polynomial is precisely that of $(a_{ij})^\sigma$. □

The corresponding statement for formal L-series follows from the local statements, by rewriting the infinite product of local Euler factors into a power series with coefficients indexed by effective Weil divisors, and applying rationality.

Theorem 8. Let ρ be an ℓ-adic representation over a global function field $K = \mathbb{F}_q(C)$ that is unramified almost everywhere, such that $\mathcal{L}_{K_v}(\rho, T)$ has coefficients in $\mathbb{Q}$ for each place v of K, and let τ be an Artin representation over K. Then $\mathcal{L}(\rho \otimes \tau, T) \in \mathbb{Q}(\tau)(T)$ and $\mathcal{L}(\rho \otimes \tau, T)^\sigma = \mathcal{L}(\rho \otimes \tau^\sigma, T)$ for any $\sigma \in G_\mathbb{Q}$. Furthermore, if $(\rho \otimes \tau)^{G_K^g} = 0$, then $\mathcal{L}(\rho \otimes \tau, T) \in \mathbb{Q}(\tau)[T]$ of degree

$$\deg \mathcal{L}(\rho \otimes \tau, T) = (2g_C - 2) \dim \rho \dim \tau + \deg \mathfrak{f}_{\rho \otimes \tau}.$$

Proof. For each place v of K, let $a_{v,n}(\tau)$ denote the coefficients of the power series $\mathcal{L}_{K_v}(\rho \otimes \tau, T)^{-1}$ for all $n \in \mathbb{N}$. By Lemma 4 for $P(T) = \mathcal{L}_{K_v}(\rho \otimes \tau, T)^{-1}$, Proposition 7 translates into $a_{v,n}(\tau) \in \mathbb{Q}(\tau)$ and $a_{v,n}(\tau)^\sigma = a_{v,n}(\tau^\sigma)$ for any $\sigma \in G_\mathbb{Q}$. Now for an effective Weil divisor $D = \sum_v n_v[v]$ on C, let $a_D(\tau)$ denote the finitely-supported product $\prod_v a_{v,n_v}(\tau)$, so that $a_D(\tau) \in \mathbb{Q}(\tau)$ and $a_D(\tau)^\sigma = a_D(\tau^\sigma)$ for any $\sigma \in G_\mathbb{Q}$. A rearrangement gives

$$\mathcal{L}(\rho \otimes \tau, T) = \prod_v \sum_{n=0}^\infty a_{v,n}(\tau)\, T^{n \deg v} = \sum_{m=0}^\infty \sum_D a_D(\tau)\, T^m,$$

where the sum ranges over effective Weil divisors D on C of degree precisely m. This is a finite sum, so that $\sum_D a_D(\tau) \in \mathbb{Q}(\tau)$ and $(\sum_D a_D(\tau))^\sigma = \sum_D a_D(\tau^\sigma)$, which proves the second statement and that $\mathcal{L}(\rho \otimes \tau, T) \in \mathbb{Q}(\tau)[[T]]$. The first and final statements follow from Proposition 5 that $\mathcal{L}(\rho \otimes \tau, T) \in \overline{\mathbb{Q}_\ell}(T)$, using the theory of Hankel determinants [Bou03, Chapter IV.4, Exercise 1]. □

In particular, these apply to $\rho = \rho_A$, which proves Theorem 2, but also to the trivial representation $\rho = \mathbb{1}$, which proves Theorem 3.

Remark 9. Using Proposition 5, Burns–Kakde–Kim proves the algebraicity and Galois equivariance of $L^{(n)}(A, \tau, s)$ up to finitely many local Euler factors away from an open set U of C [BKK18, Proposition 2.2], by directly arguing that the action of Fr_q is preserved under an isomorphism

$$H^i_{\text{ét},c}(\overline{U}, \mathcal{F}_{\rho_A} \otimes \mathcal{F}^\sigma_\tau) \cong H^i_{\text{ét},c}(\overline{U}, \mathcal{F}_{\rho_A} \otimes \mathcal{F}_\tau)^\sigma,$$

for any $\sigma \in G_{\mathbb{Q}}$, where both sides are compactly-supported étale cohomology groups of the base change $\overline{U}$ of U to $\overline{\mathbb{F}_q}$. The remaining finitely many local Euler factors can be handled separately by Proposition 7, which gives an alternative proof for Theorem 2 independent from Theorem 8.

Remark 10. There are explicit bounds for $\deg \mathfrak{f}_{\rho\otimes\tau}$ in terms of $\deg \mathfrak{f}_\rho$ and $\deg \mathfrak{f}_\tau$, such as in the arguments of Bisatt–Paterson [BP23, Section 2], so the computation of $\deg \mathcal{L}(\rho \otimes \tau, T)$ generalises that by Comeau-Lapointe–David–Lalin–Li for Dirichlet twists of elliptic curves [CLDLL22, Theorem 2.2].

The expression for $\deg \mathcal{L}(\rho \otimes \tau, T)$ in Theorem 8 is useful for computing formal L-series of Dirichlet twists of elliptic curves, which was done explicitly by Comeau-Lapointe–David–Lalin–Li using the functional equation [CLDLL22, Section 5.1]. The reader is referred to Ulmer's notes [Ulm11, Lecture 1] and Rosen's book [Ros02, Chapter 4 and Chapter 9] for the general theory over global function fields of elliptic curves and Dirichlet characters respectively.

Example 11. Let $K = \mathbb{F}_{11}(t)$, and let A be the elliptic curve over K given by $Y^2 = X^3 + (t+1)^3(t+2)^3$. Consider the Dirichlet character τ over K of modulus t given by $2 \mapsto \zeta_5$ and the automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$ given by $\zeta_5 \mapsto \zeta_5^2$. To verify Theorem 2 that $\mathcal{L}(A, \tau, T)^\sigma = \mathcal{L}(A, \tau^\sigma, T)$, first compute

$$\mathcal{L}(A, \tau, T) = 1 + 11(5 + 12\zeta_5 + 5\zeta_5^2)\, T^2 + 14641\zeta_5^2 T^4.$$

Since $\tau^\sigma = \sigma \circ \tau$ is given by $2 \mapsto \zeta_5^2$, separately compute

$$\mathcal{L}(A, \tau^\sigma, T) = 1 + 11(5 + 12\zeta_5^2 + 5\zeta_5^4)\, T^2 + 14641\zeta_5^4 T^4.$$

These were computed in Magma, but the same algorithm works in any software with support for irreducible polynomials over finite fields. Note that $\mathfrak{f}_{\rho_A} = 2[t+1]+2[t+2]$ and $\mathfrak{f}_\tau = [t]+[1/t]$ have disjoint support, so that $(\rho_A \otimes \tau)^{I_{K_t}} = 0$. In particular, $(\rho_A \otimes \tau)^{G^g_K} = 0$, so that $\mathcal{L}(A, \tau, T) \in \mathbb{Q}(\zeta_5)[T]$ of degree

$$\deg \mathcal{L}(A, \tau, T) = (2g_{\mathbb{P}^1} - 2) \dim \rho_A \dim \tau + \deg \mathfrak{f}_\tau \dim \rho_A + \deg \mathfrak{f}_{\rho_A} \dim \tau = 4.$$

Thus $\mathcal{L}(A, \tau, T)$ is completely determined by $\mathcal{L}_{K_v}(A, \tau, T)$ for all places v of K with $\deg v \leq 4$, where $\mathcal{L}_{K_v}(A, \tau, T) = 1$ for $v \in \{1/t, t, t+1, t+2\}$ and

$$\mathcal{L}_{K_v}(A, \tau, T) = 1 - \mathrm{tr}(\mathrm{Fr}_{K_v} \mid \rho_A)\, \tau(v)\, T + 11^{\deg v} \tau(v)^2\, T^2,$$

for all other places v of K. For instance, if $v = t^4 + t + 2$, then $\mathrm{tr}(\mathrm{Fr}_{K_v} \mid \rho_A) = -242$ and $\tau(v) = \zeta_5$, so that $\mathcal{L}_{K_v}(A, \tau, T) = 1 + 242\zeta_5 T + 14641\zeta_5^2 T^2$.

Remark 12. The existence of a globally compatible functional equation can drastically cut down the number of computations of local Euler factors necessary to completely determine the formal L-series. This in turn involves computing Langlands–Deligne local constants, which will not be explored in this paper.

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