Adèles and cohomology

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Class field theory

Thursday, 4 July 2024

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Theorem (Albert–Brauer–Hasse–Noether) Let K be a number field. Then there is a short exact sequence

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Why is this the fundamental exact sequence of class field theory?

In fact, it suffices to understand

$$0 \to \operatorname{Br}(L/K) \to \bigoplus_{v} \operatorname{Br}(L_{w}/K_{v}) \to \frac{1}{\#G}\mathbb{Z}/\mathbb{Z} \to 0,$$

where L/K is a finite cyclic extension with Galois group G.

Recall that for a modulus $\mathfrak{m} = \mathfrak{m}_0 \mathfrak{m}_\infty$ of a number field K,

- ▶ $I_{\mathcal{K}}(\mathfrak{m})$ is the ideal group of fractional ideals coprime to \mathfrak{m}_0 , and
- ▶ $P_{\mathcal{K}}(\mathfrak{m})$ is the ray subgroup of principal fractional ideals $\langle \alpha \rangle$ such that $\operatorname{ord}_{\mathfrak{p}}(\alpha 1) \ge \operatorname{ord}_{\mathfrak{p}}(\mathfrak{m})$ for all $\mathfrak{p} \mid \mathfrak{m}_0$ and $\sigma(\alpha) > 0$ for all $\sigma \mid \mathfrak{m}_{\infty}$.

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Theorem (global reciprocity)

Let L/K be a finite abelian extension of number fields with Galois group *G*. Then there is a surjective global Artin map

 $\Phi_{L/K}: I_K(\mathfrak{m})/P_K(\mathfrak{m}) \twoheadrightarrow G,$

with kernel precisely $Nm(I_L(\mathfrak{m}))$, where \mathfrak{m} consists of all ramified primes.

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Theorem (local reciprocity)

Let L_w/K_v be a finite abelian extension of non-archimedean local fields with Galois group G_v . Then there is a surjective **local Artin map**

$$\phi_{L_w/K_v}:K_v^\times\twoheadrightarrow G_v,$$

with kernel precisely $Nm(L_w^{\times})$.

Idèles

The **idèle group** of K is defined by

$$\mathcal{I}_{\mathcal{K}} := \left\{ (a_{v})_{v} \in \prod_{v} \mathcal{K}_{v}^{\times} \; \middle| \; a_{v} \in \mathcal{O}_{v}^{\times} \text{ for almost all } v \right\}.$$

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It is a topological group under the restricted product topology, where a basis of open sets is given by the open sets of the product

$$\prod_{v\in S} K_v^{\times} \times \prod_{v\notin S} \mathcal{O}_v^{\times},$$

where S is a finite set of places of K containing the archimedean places.

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There is a natural diagonal embedding $\Delta : \mathcal{K}^{\times} \hookrightarrow \mathcal{I}_{\mathcal{K}}$, whose image is the **principal idèle subgroup**, and whose cokernel is the **idèle class group**

$$\mathcal{C}_{\mathcal{K}} := \mathcal{I}_{\mathcal{K}} / \Delta(\mathcal{K}^{\times}).$$

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Note that $\Psi_{L/K}(a_v) = \operatorname{Fr}_v^{-\operatorname{ord}_v(a_v)}$ for all unramified places v of K.

Example ($K = \mathbb{Q}$ and $L = \mathbb{Q}(\zeta_{15})$)

There is an isomorphism of topological groups

$$egin{array}{cccc} \mathcal{I}_{\mathbb{Q}} & \stackrel{\sim}{\longrightarrow} & \mathbb{Q}^{ imes} & imes & \prod_{p} \mathbb{Z}_{p}^{ imes} \ (a_{\infty}, a_{2}, a_{3}, a_{5}, \dots) & \longmapsto & rac{a_{\infty}}{|a_{\infty}|} d & |a_{\infty}| & (rac{a_{2}}{d}, rac{a_{3}}{d}, rac{a_{5}}{d}, \dots) \end{array},$$

where $d := \prod_{p} p^{\operatorname{ord}_{p}(a_{p})}$.

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 $\operatorname{Gal}(\mathbb{Q}(\zeta_{15})/\mathbb{Q}) \xleftarrow{} (\mathbb{Z}/15\mathbb{Z})^{\times} \xleftarrow{} (\mathbb{Z}/3\mathbb{Z})^{\times} \times (\mathbb{Z}/5\mathbb{Z})^{\times}$

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$$\begin{split} \mathcal{I}_{\mathbb{Q}} & \xrightarrow{\sim} & \mathbb{Q}^{\times} & \times & \mathbb{R}^{+} & \times & \prod_{p} \mathbb{Z}_{p}^{\times} \\ (a_{\infty}, a_{2}, a_{3}, a_{5}, \ldots) & \longmapsto & \frac{a_{\infty}}{|a_{\infty}|} d & |a_{\infty}| & \left(\frac{a_{2}}{d}, \frac{a_{3}}{d}, \frac{a_{5}}{d}, \ldots\right) \\ \text{where } d &:= \prod_{p} p^{\operatorname{ord}_{p}(a_{p})}. \text{ This induces:} \\ \mathcal{C}_{\mathbb{Q}} & \xrightarrow{\sim} & \mathbb{R}^{+} \times \prod_{p} \mathbb{Z}_{p}^{\times} \longrightarrow \mathbb{Z}_{3}^{\times} \times \mathbb{Z}_{5}^{\times} \longrightarrow (\mathbb{Z}_{3}/3\mathbb{Z}_{3})^{\times} \times (\mathbb{Z}_{5}/5\mathbb{Z}_{5})^{\times} \end{split}$$

$$\begin{array}{c} \Psi_{\mathbb{Q}(\zeta_{15})/\mathbb{Q}} & \swarrow \\ \operatorname{Gal}(\mathbb{Q}(\zeta_{15})/\mathbb{Q}) \xleftarrow{} (\mathbb{Z}/15\mathbb{Z})^{\times} \xleftarrow{} (\mathbb{Z}/3\mathbb{Z})^{\times} \times (\mathbb{Z}/5\mathbb{Z})^{\times} \end{array}$$

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The idèlic Artin map $\Psi_{\mathbb{Q}(\zeta_{15})/\mathbb{Q}} : \mathcal{C}_{\mathbb{Q}} \to \operatorname{Gal}(\mathbb{Q}(\zeta_{15})/\mathbb{Q})$ maps the idèle class $[(1, 2, 1, 1, \ldots)]$ to the automorphism $\zeta_{15} \mapsto \zeta_{15}^{1/2}$.

There is a surjective **content map** $\tilde{c} : \mathcal{I}_K \twoheadrightarrow I_K$ that maps an idèle $(a_v)_v$ to the ideal $\prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(a_\mathfrak{p})}$, which descends to a surjection $c : \mathcal{C}_K \twoheadrightarrow I_K/P_K$.

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Lemma

Let G be a finite abelian group, and let \mathfrak{m} be a modulus of a number field K. Then any homomorphism $\Phi_K : I_K(\mathfrak{m}) \to G$ induces a unique continuous homomorphism $\Psi_K : \mathcal{C}_K \to G$ such that

$$\Psi_{\mathcal{K}}((a_{v})_{v}) = \Phi_{\mathcal{K}}(c((a_{v})_{v})),$$

for any $(a_v)_v \in \mathcal{I}_K$ such that $a_v = 1$ for all $v \mid \mathfrak{m}$.

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for any $(a_v)_v \in \mathcal{I}_K$ such that $a_v = 1$ for all $v \mid \mathfrak{m}$. Furthermore, any continuous homomorphism $\Psi_K : \mathcal{C}_K \to G$ arises in such a way. Since Ψ_K is a homomorphism, it is determined by idèles of the form

$$(\ldots, 1, 1, \frac{a}{v}, 1, 1, \ldots),$$

where a is either a unit or a uniformiser if v is non-archimedean.

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$$\Psi_{\mathbb{Q}}(\{up\}) = \Phi_{\mathbb{Q}}(c(\{up\})) = \Phi_{\mathbb{Q}}(p^{\operatorname{ord}_{p}(up)}) = \Phi_{\mathbb{Q}}(p) = (\zeta_{3} \mapsto \zeta_{3}^{p}).$$

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 $\Delta(\frac{a}{|a|}) \cdot \{a\} = (\dots, \frac{a}{|a|}, \frac{a}{|a|}, \frac{a}{|a|}, \frac{a}{|a|}, \frac{a}{|a|}, \dots),$

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and \mathbb{R}^+ is connected while $\operatorname{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q})$ is discrete. • Let v = 3 and a = 3. Then $\Psi_{\mathbb{Q}}(\{3\}) = \Psi_{\mathbb{Q}}(\Delta(\frac{1}{3}) \cdot \{3\}) = 1$, since $\Delta(\frac{1}{3}) \cdot \{3\} = (\dots, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots),$

and $\Psi_{\mathbb{Q}}(\{1\}) = 1$.

Example $(\Phi_{\mathbb{Q}} : I_{\mathbb{Q}}(3\infty) \to \operatorname{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}))$ For brevity, denote $\{a\} := (\dots, 1, 1, a, 1, 1, \dots).$

• Let
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• Let v = 3 and a = 2. It suffices to find a prime $p \in \mathbb{Z}$ such that

$$\Delta(p) \cdot \{2\} = (\dots, p, p, 2p, \frac{1}{3}, p, p, \dots) \cdot (\dots, 1, 1, p, 1, 1, \dots),$$

and that $2p \to 1$ in \mathbb{Z}_3 , so that $\Psi_{\mathbb{Q}}(\{2p\}) = 1$ by continuity.

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$$\Psi_{\mathbb{Q}}(\{2\}) = \Phi_{\mathbb{Q}}(p) = (\zeta_3 \mapsto \zeta_3^p) = (\zeta_3 \mapsto \zeta_3^2),$$

which does not depend on p.

Example $(\Phi_{\mathbb{Q}} : I_{\mathbb{Q}}(3\infty) \to \operatorname{Gal}(\mathbb{Q}(\zeta_3)/\mathbb{Q}))$ For brevity, denote $\{a\} := (\dots, 1, 1, a, 1, 1, \dots).$

• Let v = 3 and a = 2. It suffices to find a prime $p \in \mathbb{Z}$ such that

$$\Delta(p) \cdot \{2\} = (\dots, p, p, 2p, \frac{1}{3}, p, p, \dots) \cdot (\dots, 1, 1, p, 1, 1, \dots),$$

and that 2p o 1 in \mathbb{Z}_3 , so that $\Psi_{\mathbb{Q}}(\{2p\}) = 1$ by continuity. Then

$$\Psi_{\mathbb{Q}}(\{2\}) = \Phi_{\mathbb{Q}}(p) = (\zeta_3 \mapsto \zeta_3^p) = (\zeta_3 \mapsto \zeta_3^2),$$

which does not depend on p. Now $\frac{1}{2}=2+\sum_{i=1}^{\infty}3^{i}$ in $\mathbb{Z}_{3},$ so set

$$p := 2 + \sum_{i=1}^{15} 3^i = 21523361,$$

which is prime in \mathbb{Z} , and $2p = 1 + 3^{16} \rightarrow 1$ in \mathbb{Z}_3 .

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Let G be a finite group, and let M be a G-module. Recall that group cohomology $H^i(G, -)$ is the right derived functor of $(-)^G$, where

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A short exact sequence of *G*-modules $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ induces a long exact sequence of cohomology groups

$$\cdots \to H^1(G,B) \xrightarrow{\overline{g}} H^1(G,C) \xrightarrow{\delta} H^2(G,A) \xrightarrow{\overline{f}} H^2(G,B) \to \ldots$$

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Note that \mathbb{Q} is torsion-free and divisible, so $H^i(G, \mathbb{Q}) = 0$ for all i > 0. In particular, there is an isomorphism $\delta : H^1(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} H^2(G, \mathbb{Z})$.

Theorem (Tate)

Let *M* be a *G*-module, such that for all subgroups $H \leq G$, T1 $H^1(H, M) = 0$, and T2 $H^2(H, M)$ is cyclic of order #*H*. Then there is an explicit isomorphism $G^{ab} \xrightarrow{\sim} M^G/\operatorname{Nm}(M)$.

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This is the key result in abstract class field theory.

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▶ If $G = \operatorname{Gal}(L_w/K_v)$ and $M = L_w^{\times}$, this gives the local reciprocity law

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▶ If G = Gal(L/K) and $M = C_L$, this gives the global reciprocity law

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Theorem (local class field theory)

Let L_w/K_v be a finite unramified extension of non-archimedean local fields with Galois group G_v . Then $H^i(G_v, \mathcal{O}_w^{\times}) = 0$ for all i > 0.

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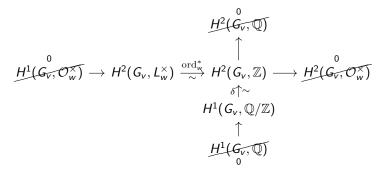
The short exact sequence $1 \to \mathcal{O}_w^{\times} \to L_w^{\times} \xrightarrow{\operatorname{ord}_w} \mathbb{Z} \to 0$ induces:

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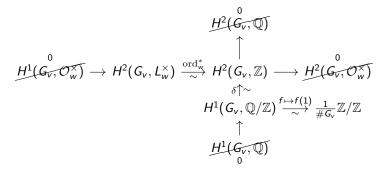


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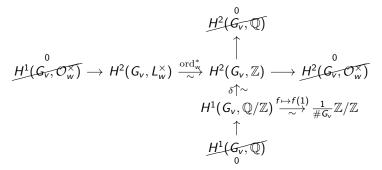


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In particular, T2 holds for L_w^{\times} .

The local invariant map is $\operatorname{inv}_{v}: H^{2}(G_{v}, L_{w}^{\times}) \to \frac{1}{\#G_{v}}\mathbb{Z}/\mathbb{Z}.$

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Example $(K_v = \mathbb{Q}_2 \text{ and } L_w = \mathbb{Q}_2(\zeta_7))$ Note that $G_v = \{1, \sigma, \sigma^2\}$, so that $\frac{1}{\#G_v}\mathbb{Z}/\mathbb{Z} = \{[0], [\frac{1}{3}], [\frac{2}{3}]\}.$

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After choosing a lift and applying δ ,

- $\delta(f_0)$ is the trivial 2-cocycle,
- $\delta(f_1)$ maps (g, h) to 1 iff $(g, h) = (\sigma, \sigma^2), (\sigma^2, \sigma), (\sigma^2, \sigma^2)$, and
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Since $\mathbb{Q}_2(\zeta_7)^{\times} \cong \mathbb{Z}_2[\zeta_7]^{\times} \times 2^{\mathbb{Z}}$,

- ▶ inv₂⁻¹[0] is the trivial 2-cocycle,
- $\operatorname{inv}_2^{-1}[\frac{1}{3}]$ maps (g, h) to 2 iff $(g, h) = (\sigma, \sigma^2), (\sigma^2, \sigma), (\sigma^2, \sigma^2)$, and
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Theorem (global class field theory)

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Thus there are inequalities

$$\#\operatorname{coker}(\overline{\Delta}) \leq \#H^2(G, \mathcal{C}_L) \leq \#G,$$

where the right inequality is an equality if G is cyclic.

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1. There are canonical isomorphisms $H^i(G, \mathcal{I}_L) \cong \bigoplus_{v} H^i(G_v, L_w^{\times})$ for all i > 0. In particular, there is an idèlic invariant map

$$\sum_{\nu} \operatorname{inv}_{\nu} : H^2(G, \mathcal{I}_L) \to \frac{1}{\operatorname{lcm}_{\nu}(\#G_{\nu})}\mathbb{Z}/\mathbb{Z}.$$

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2. If $a \in H^2(G, L^{\times})$, then $\sum_{v} inv_v(a) = 0$.

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If a ∈ H²(G, L[×]), then ∑_v inv_v(a) = 0.
 If G is cyclic, then ∑_v inv_v surjects onto ¹/_{#G}Z/Z.

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Proof.

- $1. \ \mbox{Follows}$ from the cohomology of unramified units.
- 2. Follows from the product formula and explicit description of $\mathrm{inv}_{\nu}.$
- 3. Follows from Chebotarev's density theorem and surjectivity of inv_{ν} .

In summary, if G is cyclic, there is a chain complex

$$0 \to H^2(G, L^{\times}) \xrightarrow{\overline{\Delta}} H^2(G, \mathcal{I}_L) \xrightarrow{\sum_v \operatorname{inv}_v} \frac{1}{\#G} \mathbb{Z}/\mathbb{Z} \to 0,$$

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On the other hand, recall that

$$\operatorname{Br}(L/K) = H^2(G, L^{\times}), \qquad \operatorname{Br}(L_w/K_v) = H^2(G_v, L_w^{\times}).$$

In summary, if G is cyclic, there is a chain complex

$$0 o H^2(G, L^{ imes}) \xrightarrow{\overline{\Delta}} H^2(G, \mathcal{I}_L) \xrightarrow{\sum_{v} \operatorname{inv}_{v}} \frac{1}{\#G} \mathbb{Z} / \mathbb{Z} o 0,$$

which is exact except possibly at the middle. However, it is also exact by

$$\#G \leq \#\operatorname{coker}(\overline{\Delta}) \leq \#H^2(G, \mathcal{C}_L) = \#G.$$

On the other hand, recall that

$$\operatorname{Br}(L/K) = H^2(G, L^{\times}), \qquad \operatorname{Br}(L_w/K_v) = H^2(G_v, L_w^{\times}).$$

This proves that the sequence

$$0 \to \operatorname{Br}(L/K) \xrightarrow{\overline{\Delta}} \bigoplus_{v} \operatorname{Br}(L_{w}/K_{v}) \xrightarrow{\sum_{v} \operatorname{inv}_{v}} \frac{1}{\#G} \mathbb{Z}/\mathbb{Z} \to 0$$

is exact.