Algebraising foundations of elliptic curves

David Kurniadi Angdinata
(joint work with Junyan Xu)

London School of Geometry and Number Theory

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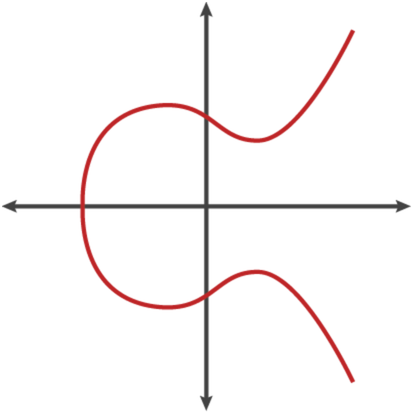
Introduction

Elliptic curves are algebraic curves given by cubic equations.



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They see many computational applications.

- Intractability of the *discrete logarithm problem* for elliptic curves forms the basis behind many public key cryptographic protocols.
- The *Atkin–Morain primality test* and *Lenstra's factorisation method* use elliptic curves and are two of the fastest known algorithms.

Formalising the theory of elliptic curves would be great!

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- Junyan Xu and I (2024) discovered gaps in the standard proof of the *multiplication-by-n formula* on an elliptic curve, filled them in with *novel* arguments, and formalised the entire proof in Lean.

Elliptic curves

An **elliptic curve** over a field F is a smooth projective curve E over F of genus one, equipped with a distinguished point $\mathcal{O}$ defined over F.

These are all notions from modern algebraic geometry.

- A **curve** is a variety [1] of dimension one as a topological space.
- **Projective** means there is a closed immersion $E \hookrightarrow \mathrm{Proj}(F[X_i])$.
- **Smooth** essentially means all $\mathcal{O}_{E,\overline{x}}$ are regular local rings.
- **Genus** is the dimension of $H^1(E, \mathcal{O}_E)$ as an F-vector space.

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Thanks to the work of Andrew Yang, Christian Merten, Joël Riou, and others, we can *almost* formalise the definition of elliptic curves in Lean!

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Elliptic curves in `mathlib`

Corollary (of the Riemann–Roch theorem)

The set of points of an elliptic curve over F is the vanishing locus of

$$\mathcal{E} := Y^2 + a_1XY + a_3Y - (X^3 + a_2X^2 + a_4X + a_6),$$

for some $a_i \in F$ such that $\Delta \neq 0$, [2] *with an extra point at infinity $\mathcal{O}$.*

[2] $\Delta := -(a_1^2+4a_2)^2(a_1^2a_6+4a_2a_6-a_1a_3a_4+a_2a_3^2-a_4^2)-8(2a_4+a_1a_3)^3-27(a_3^2+4a_6)^2+9(a_1^2+4a_2)(2a_4+a_1a_3)(a_3^2+4a_6)$

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The arithmetic can be formalised independently of the algebraic geometry.

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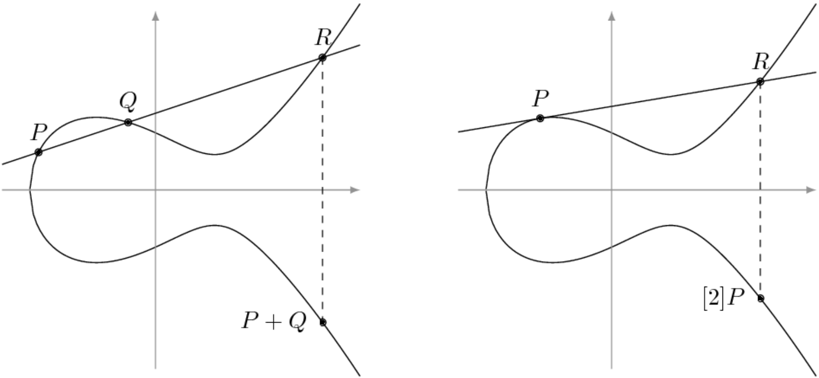
The group law

The set of points $E(F)$ can be endowed with a geometric addition law.



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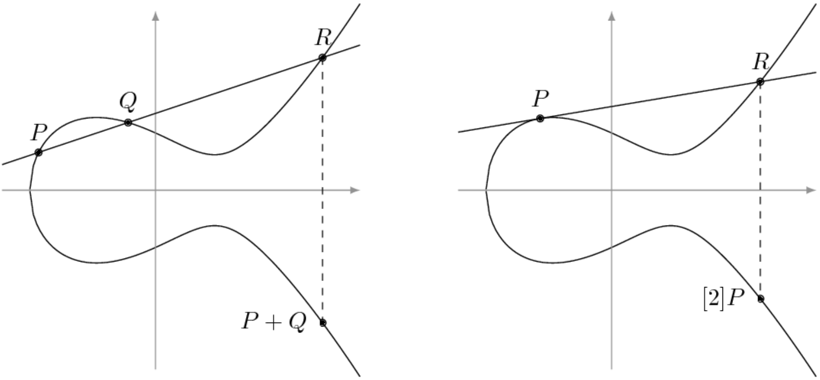

Theorem (the group law)

This addition law makes $E(F)$ an abelian group with identity $\mathcal{O}$.

The group law in `mathlib`

In `mathlib`, the addition law is given by explicit rational functions.

For instance, $(x_1, y_1) + (x_2, y_2) := (x_3, y_3)$, where

$$x_3 := \lambda^2 + a_1\lambda - a_2 - x_1 - x_2,$$
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Here, the slope λ is given by

$$\lambda := \begin{cases} \dfrac{y_1 - y_2}{x_1 - x_2} & x_1 \neq x_2 \\ \dfrac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x + a_3} & y_1 \neq -y_1 - a_1x - a_3 \\ \infty & \text{otherwise} \end{cases}.$$

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All of the axioms for an abelian group are easy *except for associativity*.

Associativity

Associativity is the statement that, for all $P, Q, R \in E(F)$,

$$(P + Q) + R = P + (Q + R).$$

In the generic case, [3] checking that their X-coordinates are equal is an equality of multivariate polynomials with 26,082 terms!

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When $\mathrm{char}(F) \neq 2, 3$, a linear change of variables reduces $\mathcal{E}$ to

$$\mathcal{E}' := Y^2 - (X^3 + aX + b),$$

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Automation in an interactive theorem prover enables manipulation of multivariate polynomials with at most 5,000 terms.

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A complex uniformisation

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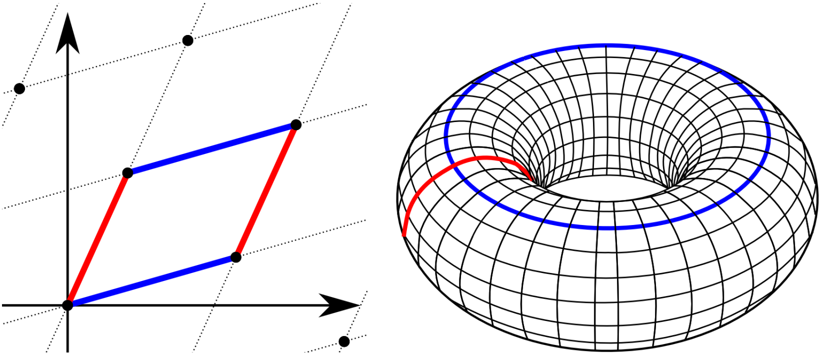
Over $F = \mathbb{C}$, an elliptic curve is just a complex torus $\mathbb{C}/\Lambda_E$.



There is an explicit bijection from $E(\mathbb{C})$ to $\mathbb{C}/\Lambda_E$ that preserves the addition law, so the group law on $\mathbb{C}/\Lambda_E$ can be pulled back to $E(\mathbb{C})$.

An algebraic variant

In general, Riemann–Roch gives an explicit bijection from $E(F)$ to the degree-zero divisor class group $\mathrm{Pic}^0_F(E)$ that preserves the addition law.

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While `mathlib` does not have divisors, it has ideals of integral domains D and the ideal class group [4] $\mathrm{Cl}(D)$, which are *purely commutative algebra*.

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The map $E(F) \to \mathrm{Pic}_F^0(E)$ translates to the map

$$\begin{array}{rcl} E(F) & \longrightarrow & \mathrm{Cl}(D) \\ \mathcal{O} & \longmapsto & [\langle 1 \rangle] \\ (x, y) & \longmapsto & [\langle X - x, Y - y \rangle] \end{array},$$

where D is the integral domain $F[X, Y]/\langle \mathcal{E} \rangle$.

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Theorem (Xu)

Proving that this map is injective only needs linear algebra.

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Multiplication by n

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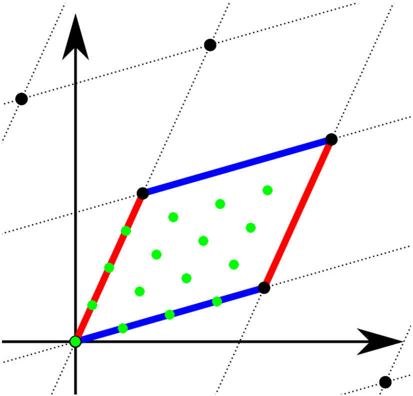
How many points $P \in E(\mathbb{C})$ are there such that $[n](P) = \mathcal{O}$?



The n-torsion subgroup

For each $n \in \mathbb{Z}$, define $E_F[n] := \{P \in E(F) : [n](P) = \mathcal{O}\}$.

Theorem (the n-torsion subgroup structure)

If $\mathrm{char}(F) \nmid n$, then $E_{\overline{F}}[n]$ is isomorphic to $(\mathbb{Z}/n)^2$.

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When $\mathrm{char}(F) \neq p$,

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When $\mathrm{char}(F) \neq p$, the p-**adic Tate module** $T_p E_{\overline{F}}$ sits in the diagram

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This makes $T_p E_{\overline{F}}$ a *two-dimensional p-adic Galois representation.* [5]

---

[5] crucial in the Mordell–Weil theorem, Tate's isogeny theorem, Serre's open image theorem, Wiles's modularity theorem, the Birch and Swinnerton-Dyer conjecture, etc.

An infamous exercise

The Arithmetic of Elliptic Curves by Silverman gives a formula for $[n](P)$.

Exercise (3.7(d))

Let $n \in \mathbb{Z}$. Prove that for any affine point $(x, y) \in E(F)$,

$$[n]((x,y)) = \left(\frac{\phi_n(x,y)}{\psi_n(x,y)^2}, \frac{\omega_n(x,y)}{\psi_n(x,y)^3} \right).$$

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This formula does not account for affine points $(x, y) \in E(F)$ such that $\psi_n(x, y) = 0$, which occurs precisely when $[n]((x, y)) = \mathcal{O}$.

Projective coordinates

In projective coordinates, the multiplication-by-n formula becomes

$$[n]((x, y)) = [(\phi_n(x, y)\psi_n(x, y), \omega_n(x, y), \psi_n(x, y)^3)].$$

In `mathlib`, a **projective point** is a class of $(x, y, z) \in F^3$ such that

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$

The point at infinity becomes $[(0, 1, 0)]$.

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The point at infinity becomes $[(0, 1, 0)]$.

More naturally, in projective coordinates with weights $(2, 3, 1)$,

$$[n]((x, y)) = [(\phi_n(x, y), \omega_n(x, y), \psi_n(x, y))].$$

In `mathlib`, a **Jacobian point** is a class of $(x, y, z) \in F^3$ such that

$$y^2 + a_1xyz + a_3yz^3 = x^3 + a_2x^2z^2 + a_4xz^4 + a_6z^6.$$

The point at infinity becomes $[(1, 1, 0)]$.

The polynomials ψ_n

For any ring R, the n**-th division polynomial** $\psi_n \in R[X, Y]$ is given by

$$\begin{aligned}
\psi_0 &:= 0, \\
\psi_1 &:= 1, \\
\psi_2 &:= 2Y + a_1 X + a_3, \\
\psi_3 &:= 3X^4 + (a_1^2 + 4a_2)X^3 + 3(2a_4 + a_1 a_3)X^2 + 3(a_3^2 + 4a_6)X + (a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2), \\
\psi_4 &:= \psi_2 \left(\begin{array}{c} 2X^6 + (a_1^2 + 4a_2)X^5 + 5(2a_4 + a_1 a_3)X^4 + 10(a_3^2 + 4a_6)X^3 + 10(a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2)X^2 \\ + ((a_1^2 + 4a_2)(a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2) - (2a_4 + a_1 a_3)(a_3^2 + 4a_6))X \\ + ((2a_4 + a_1 a_3)(a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2) - (a_3^2 + 4a_6)^2) \end{array} \right), \\
\psi_{2n+1} &:= \psi_{n+2}\psi_n^3 - \psi_{n-1}\psi_{n+1}^3, \\
\psi_{2n} &:= \frac{\psi_{n-1}^2 \psi_n \psi_{n+2} - \psi_{n-2}\psi_n \psi_{n+1}^2}{\psi_2}, \\
\psi_{-n} &:= -\psi_n.
\end{aligned}$$

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\psi_4 &:= \psi_2 \begin{pmatrix} 2X^6 + (a_1^2 + 4a_2)X^5 + 5(2a_4 + a_1 a_3)X^4 + 10(a_3^2 + 4a_6)X^3 + 10(a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2)X^2 \\ + ((a_1^2 + 4a_2)(a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2) - (2a_4 + a_1 a_3)(a_3^2 + 4a_6))X \\ + ((2a_4 + a_1 a_3)(a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2) - (a_3^2 + 4a_6)^2) \end{pmatrix}, \\
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\end{aligned}$$

In `mathlib`, ψ_n is defined in terms of some polynomial $\Psi_n \in R[X]$ such that $\psi_n = \Psi_n$ when n is odd and $\psi_n = \psi_2 \Psi_n$ when n is even.

The polynomials ϕ_n

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$$\begin{aligned}\psi_2^2 &= (2Y + a_1X + a_3)^2 \\ &= 4(Y^2 + a_1XY + a_3Y) + a_1^2X^2 + 2a_1a_3X + a_3^2 \\ &\equiv 4X^3 + b_2X^2 + 2b_4X + b_6 \mod \mathcal{E},\end{aligned}$$

so ψ_n^2 and $\psi_{n+1}\psi_{n-1}$ are congruent to polynomials in $R[X]$.

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Exercise (3.7(c))

Let $n \in \mathbb{Z}$. Prove that ϕ_n and ψ_n^2 have no common roots.

This *needs* Exercise 3.7(d) and the assumption that $\Delta \neq 0$.

The polynomials ω_n

The polynomial $\omega_n \in R[X, Y]$ is given by

$$\omega_n := \frac{1}{2}\left(\frac{\psi_{2n}}{\psi_n} - a_1\phi_n\psi_n - a_3\psi_n^3\right).$$

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Lemma (Xu)

Let $n \in \mathbb{Z}$. Then $\psi_{2n}/\psi_n - a_1\phi_n\psi_n - a_3\psi_n^3$ is divisible by 2 in $\mathbb{Z}[a_i, X, Y]$.

Example ($a_1 = a_3 = 0$)

$$\omega_2 = \frac{2X^6+4a_2X^5+10a_4X^4+40a_6X^3+10(4a_2a_6-a_4^2)X^2+(4a_2(4a_2a_6-a_4^2)-8a_4a_6)X+(2a_4(4a_2a_6-a_4^2)-16a_6^2)}{2}.$$

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Define ω_n as the image of the quotient under $\mathbb{Z}[a_i, X, Y] \to R[X, Y]$.

When $n = 4$, this quotient has 15,049 terms.

Elliptic divisibility sequences

Integrality relies on the fact that ψ_n is an **elliptic divisibility sequence**.

Exercise (3.7(g))

For all $n, m, r \in \mathbb{Z}$, prove that $\psi_n \mid \psi_{nm}$ and

$$\mathrm{ES}(n, m, r) : \psi_{n+m}\psi_{n-m}\psi_r^2 = \psi_{n+r}\psi_{n-r}\psi_m^2 - \psi_{m+r}\psi_{m-r}\psi_n^2.$$

Note that $\mathrm{ES}(n+1, n, 1)$ gives ψ_{2n+1} and $\mathrm{ES}(n+1, n-1, 1)$ gives ψ_{2n}.

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Note that $\mathrm{ES}(n+1, n, 1)$ gives ψ_{2n+1} and $\mathrm{ES}(n+1, n-1, 1)$ gives ψ_{2n}.

Surprisingly, this needs the stronger result that ψ_n is an **elliptic net**.

Theorem (Xu)

Let $n, m, r, s \in \mathbb{Z}$. Then

$$\begin{aligned}\mathrm{EN}(n, m, r, s) : \psi_{n+m}\psi_{n-m}\psi_{r+s}\psi_{r-s} = \psi_{n+r}\psi_{n-r}\psi_{m+s}\psi_{m-s} \\ - \psi_{m+r}\psi_{m-r}\psi_{n+s}\psi_{n-s}.\end{aligned}$$

Xu gave an elegant proof of this on Math Stack Exchange.

Ellipticity of ψ_n

It suffices to prove $\mathrm{EN}(n, m, r, s)$ by strong induction on n assuming that $n, m, r, s \in \mathbb{N}$ [6] such that $n > m > r > s$.

[6] the complete proof also needs the case when $n, m, r, s \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}$

Ellipticity of ψ_n

It suffices to prove $\mathrm{EN}(n, m, r, s)$ by strong induction on n assuming that $n, m, r, s \in \mathbb{N}$ [6] such that $n > m > r > s$. Firstly,

$$\mathrm{EN}(n, m, 1, 0) = \mathrm{EN}(\tfrac{n+m+1}{2}, \tfrac{n+m-1}{2}, \tfrac{n-m+1}{2}, \tfrac{n-m-1}{2}).$$

If $n = m + 1$, then $\mathrm{EN}(m + 1, m, 1, 0)$ holds by definition of ψ_{2n+1}. Otherwise $n > m + 1$, then inductive hypothesis applies since $\frac{n+m+1}{2} < n$. This gives $\mathrm{EN}(n, m, 1, 0)$ for all $n, m > 1$.

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$$\begin{aligned}
\mathrm{EN}(n,m,r,0) &= \psi_r^2 \cdot \mathrm{EN}(n,m,1,0) - \psi_m^2 \cdot \mathrm{EN}(n,r,1,0) + \psi_n^2 \cdot \mathrm{EN}(m,r,1,0),\\
\mathrm{EN}(n,m,r,1) &= \psi_{r+1}\psi_{r-1} \cdot \mathrm{EN}(n,m,1,0) - \psi_{m+1}\psi_{m-1} \cdot \mathrm{EN}(n,r,1,0) + \psi_{n+1}\psi_{n-1} \cdot \mathrm{EN}(m,r,1,0).
\end{aligned}$$

This gives $\mathrm{EN}(n, m, r, 0)$ and $\mathrm{EN}(n, m, r, 1)$ for all $n, m, r > 1$.

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If $n = m + 1$, then $\mathrm{EN}(m + 1, m, 1, 0)$ holds by definition of ψ_{2n+1}. Otherwise $n > m + 1$, then inductive hypothesis applies since $\frac{n+m+1}{2} < n$. This gives $\mathrm{EN}(n, m, 1, 0)$ for all $n, m > 1$. Furthermore,

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\mathrm{EN}(n,m,r,1) &= \psi_{r+1}\psi_{r-1} \cdot \mathrm{EN}(n,m,1,0) - \psi_{m+1}\psi_{m-1} \cdot \mathrm{EN}(n,r,1,0) + \psi_{n+1}\psi_{n-1} \cdot \mathrm{EN}(m,r,1,0).
\end{aligned}$$

This gives $\mathrm{EN}(n, m, r, 0)$ and $\mathrm{EN}(n, m, r, 1)$ for all $n, m, r > 1$. Finally,

$$\begin{aligned}
\mathrm{EN}(n,m,r,s) = \;& \psi_m^2 \cdot \mathrm{EN}(n,r,s,1) + \psi_{m+1}\psi_{m-1} \cdot \mathrm{EN}(n,r,s,0) + \psi_{m+r}\psi_{m-r} \cdot \mathrm{EN}(n,s,1,0) \\
- \;& \psi_r^2 \cdot \mathrm{EN}(n,m,s,1) - \psi_{r+1}\psi_{r-1} \cdot \mathrm{EN}(n,m,s,0) - \psi_{m+s}\psi_{m-s} \cdot \mathrm{EN}(n,r,1,0) \\
+ \;& \psi_s^2 \cdot \mathrm{EN}(n,m,r,1) + \psi_{s+1}\psi_{s-1} \cdot \mathrm{EN}(n,m,r,0) + \psi_{r+s}\psi_{r-s} \cdot \mathrm{EN}(n,m,1,0) \\
- \;& 2\psi_n^2 \cdot \mathrm{EN}(m,r,s,1) \quad .
\end{aligned}$$

This gives $\mathrm{EN}(n, m, r, s)$ for all $n, m, r, s > 1$.

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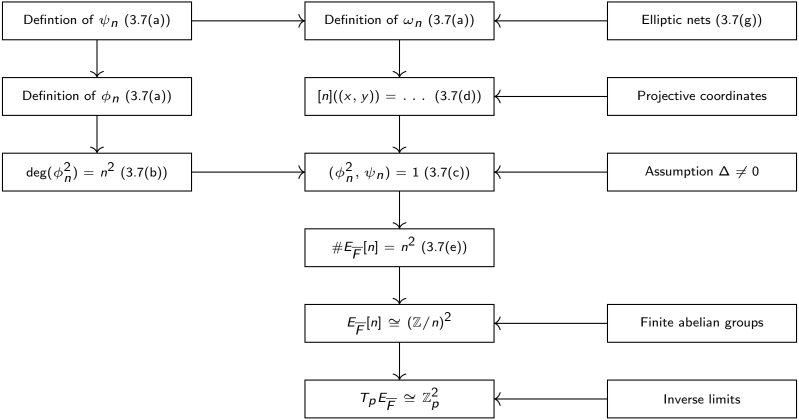
Blueprint for $T_p E_{\overline{F}}$



Future projects

Projects without algebraic geometry:

- algorithms that only use the group law
- finite fields: the Hasse–Weil bound, the Weil conjectures
- local fields: the reduction homomorphism, Tate's algorithm, the Neron–Ogg–Shafarevich criterion, the Hasse–Weil L-function
- number fields: Neron-Tate heights, the Mordell–Weil theorem, Tate–Shafarevich groups, the Birch and Swinnerton-Dyer conjecture
- complete fields: complex uniformisation, p-adic uniformisation

Projects with algebraic geometry:

- elliptic curves over global function fields
- the projective scheme associated to an elliptic curve
- integral models and finite flat group schemes
- divisors on curves and the Riemann–Roch theorem
- modular curves and Mazur's theorem