Imperial College London

# Arithmetic Statistics for Elliptic Curves

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MEng Pure Mathematics and Computational Logic

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# Some motivation

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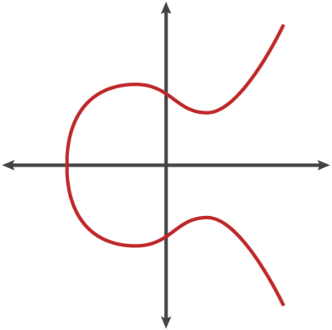

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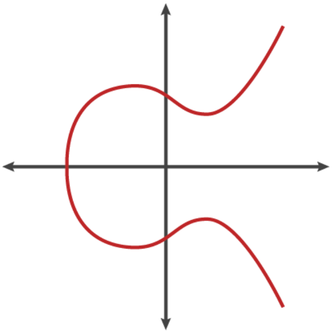

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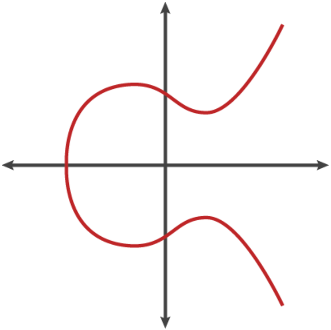

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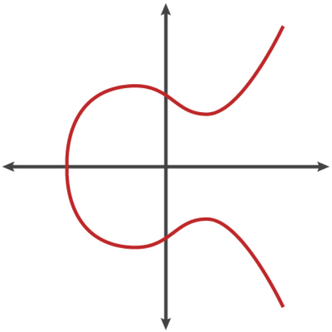

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- Number theory.
- Cryptography.

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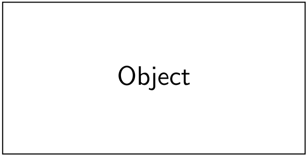
What can we do?

- Study Selmer groups and Tate-Shafarevich groups.
- Neither are easy to study.
- Study models for them instead.

# Framework and overview



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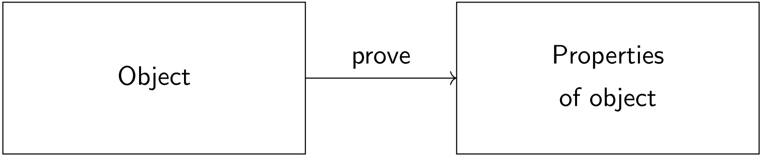

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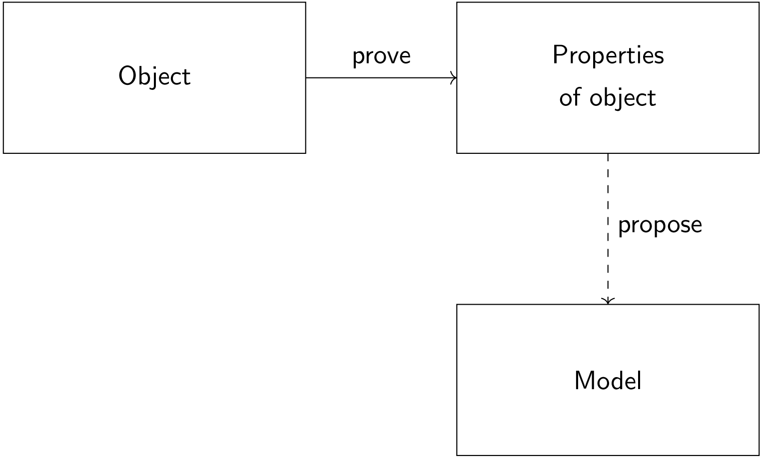

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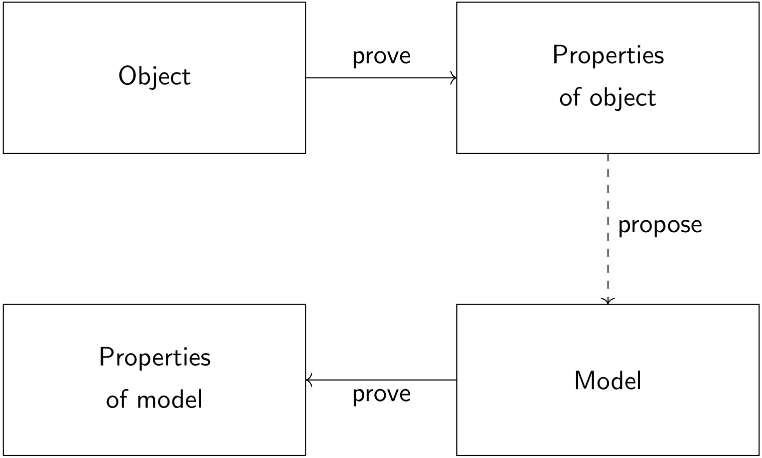

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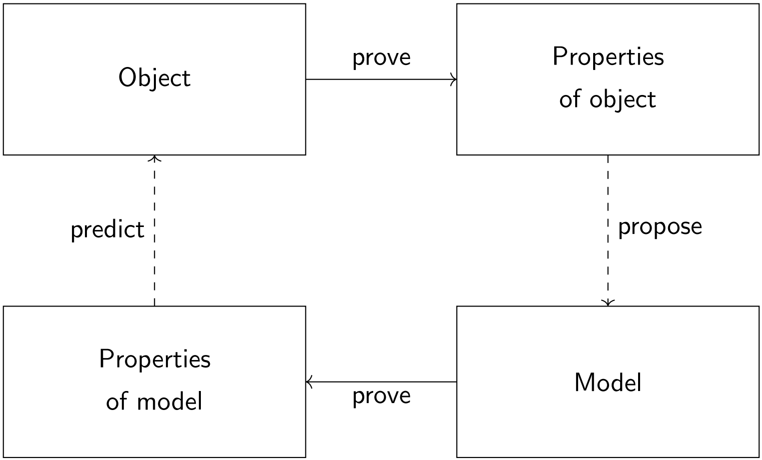

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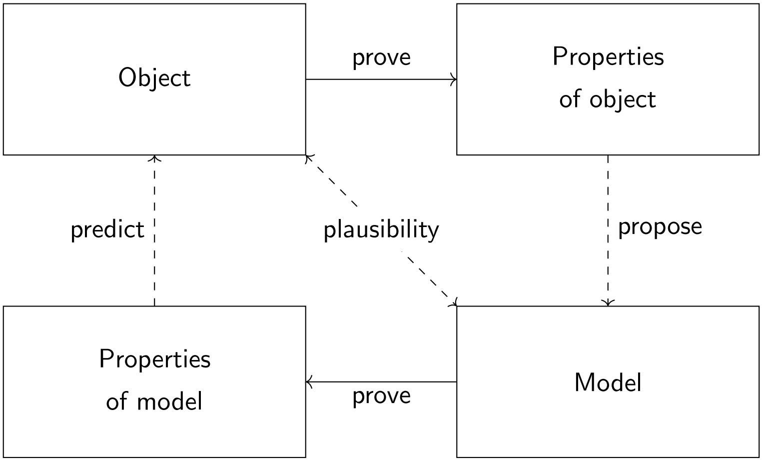

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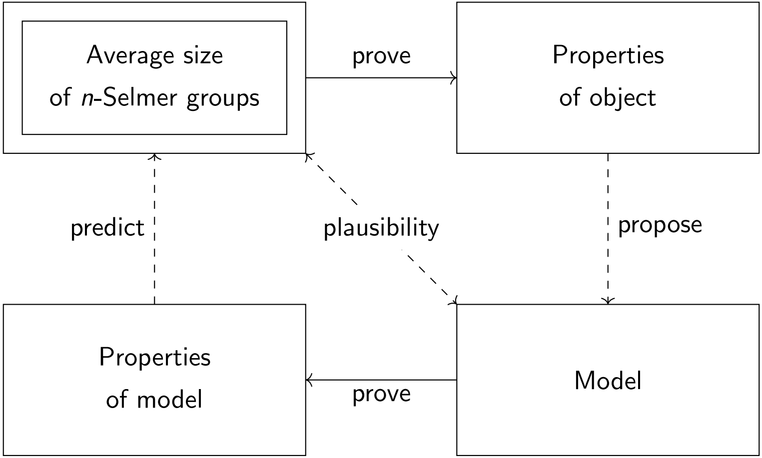

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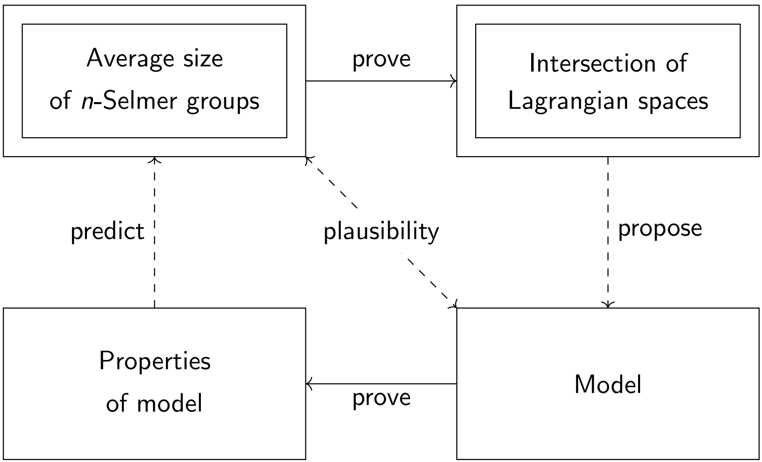

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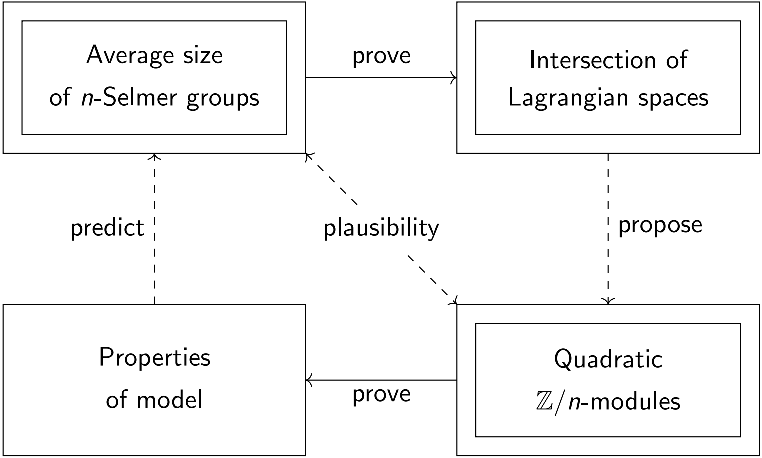

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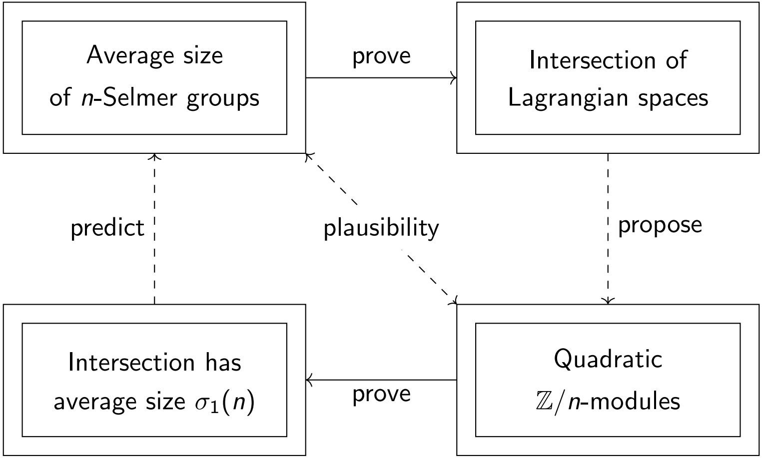

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"All but finitely many rational elliptic curves have rank at most 21"

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The **Mordell-Weil rank** is $\mathrm{rk}(E/K)$.

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Truncating at $H^1(K, E[n])$,

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## Theorem (1)

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- *Almost all*: limiting proportion when ordered by height.

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Think of $M = (\mathbb{Z}/p^e)^{2n}$, equipped with hyperbolic quadratic form

$$(x_1, \dots, x_n, y_1, \dots, y_n) \mapsto \sum_{i=1}^{n} x_i y_i,$$

with Lagrangian submodule $N = (\mathbb{Z}/p^e)^n \oplus 0^n$.

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References:

- Colliot-Thélène, Skorobogatov, Swinnerton-Dyer (1998): $p^e = 2$ and finite-dimensional construction. [1]
- Bhargava, Kane, Lenstra, Poonen, Rains (2015): general $p^e$, infinite-rank construction, and generalisations to abelian varieties with arbitrary isogenies over arbitrary global fields. [2]

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[1] J.-L. Colliot-Thelene, A. Skorobogatov and P. Swinnerton-Dyer. 'Hasse principle for pencils of curves of genus one whose Jacobians have rational 2-division points'. In: Invent. Math. 134 (1998)

[2] M. Bhargava, D. Kane, H. Lenstra, B. Poonen and E. Rains. 'Modelling the distribution of ranks, Selmer groups, and Shafarevich-Tate groups of elliptic curves'. In: Camb. J. Math. 3 (2015)

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    - Construct $\Theta$ such that $0 \to \overline{K_v}^{\times} \to \Theta \to E[n] \to 0$.

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    - Extract assumption $\mathrm{SL}_2(\mathbb{Z}/n) \leq \operatorname{im}\rho_{E[n]}$ and justify its ubiquity using Hilbert's irreducibility theorem and division polynomials. □

# Model for Selmer groups

## Theorem (2)

*The average size of the intersection of two Lagrangian direct summands of the quadratic* $\mathbb{Z}/p^e$*-module* $(\mathbb{Z}/p^e)^{2n}$ *chosen uniformly at random tends to the sum of divisors* $\sigma_1$ *of* $p^e$ *as* $n \to \infty$*.*

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Reference:

- Poonen, Rains (2012): $e = 1$. [1]

[1] B. Poonen and E. Rains. 'Random maximal isotropic subspaces and Selmer groups'. In: J. Amer. Math. Soc 25 (2012)

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   - Deduce result by telescoping argument. □

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A model for $n$-Selmer groups.

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THANK YOU