An unusual cubic representation problem

David Kurniadi Angdinata

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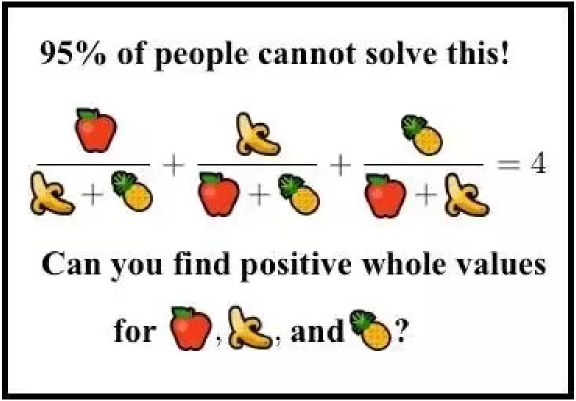

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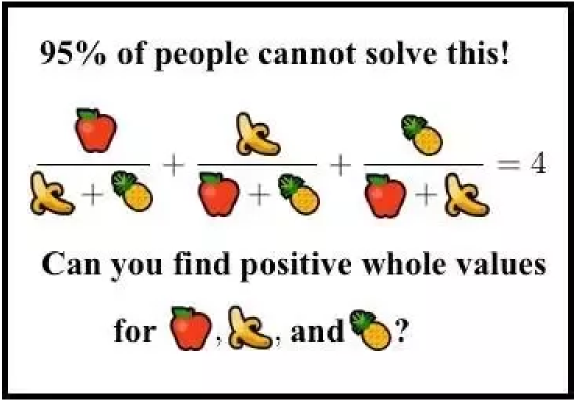
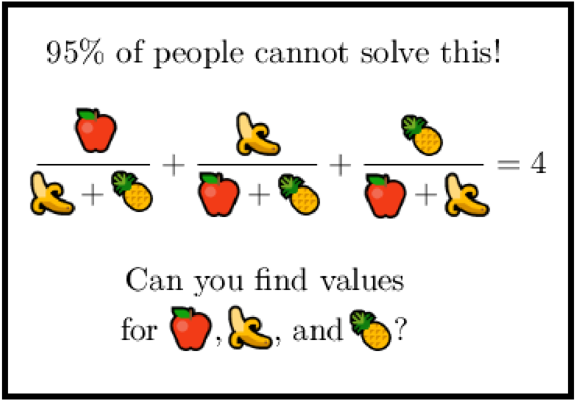


$$\text{APPLE} = 154476802108746166441951315019919837485664325669565431700026634898253202035277999$$

$$\text{BANANA} = 36875131794129999827197811565225474825492979968971970996283137471637224634055579$$

$$\text{PINEAPPLE} = 4373612677928697257861252602371390152816537558161613618621437993378423467772036$$

A trivial cubic representation problem



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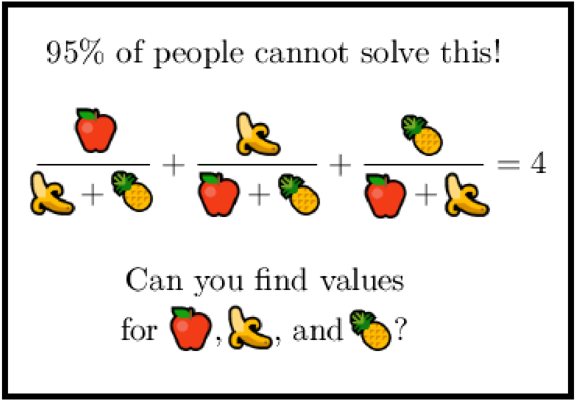

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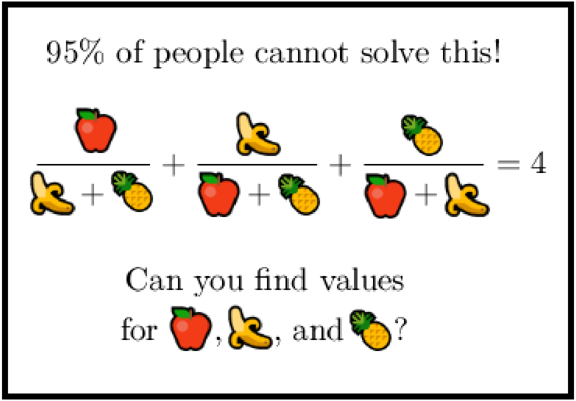

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$$\mathbb{Q}, \mathbb{Z} \quad : \quad (a, b, c) \quad = \quad (11, 4, -1)$$

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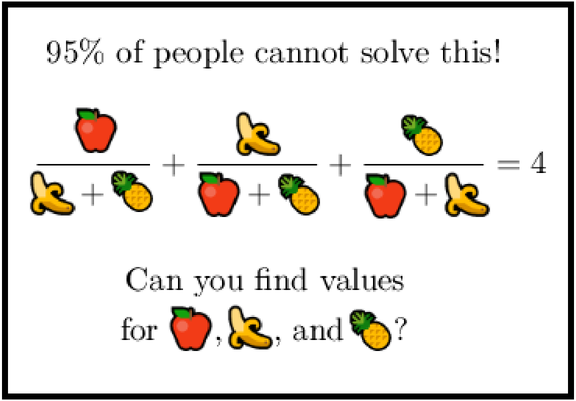

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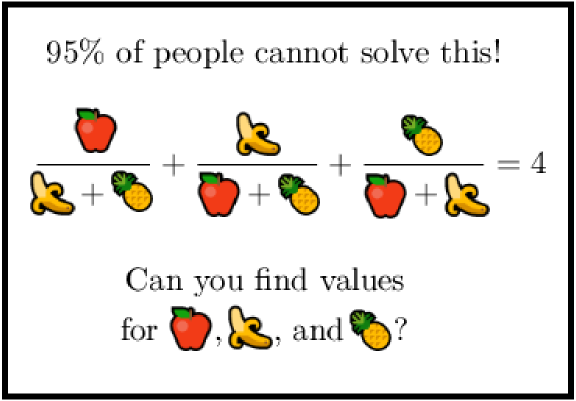

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& &&= (-11, -4, 1) \\
& &&= (1, -4, -11) \\
& &&= \dots
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A less unusual cubic representation problem

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = 4, \qquad a, b, c \in \mathbb{Z}.$$

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$$= 4(a+b)(a+c)(b+c).$$

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$$a^3 + a^2c + a^2b + abc + ab^2 + abc + b^3 + b^2c + abc + ac^2 + bc^2 + c^3$$

$$= 4a^2b + 4abc + 4a^2c + 4ac^2 + 4ab^2 + 4b^2c + 4abc + 4bc^2.$$

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- Require $a + b, a + c, b + c > 0$.

Dimensionality of solution space

$$a^3 + b^3 + c^3 - 5abc - 3\left(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2\right) = 0, \qquad a, b, c \in \mathbb{Z}$$

Definition

A polynomial is **homogeneous** if all of its monomials have the same total degree.

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If (a_0, b_0, c_0) is a solution in $\mathbb{Z}$, then $(\lambda a_0, \lambda b_0, \lambda c_0)$ is a solution in $\mathbb{Q}$ for any $\lambda \in \mathbb{Q}^*$.

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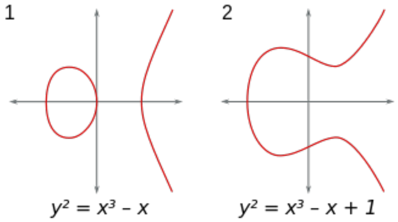
- Modulo $\sim$, the equation is cubic of two variables.

Elliptic curves: informally



An *elliptic curve* is the space of solutions to a cubic equation

$$y^2 = x^3 + Ax + B,$$

where A and B are in some field such that $4A^3 + 27B^2 \neq 0$.

- Simplest non-trivial structures in algebraic geometry.
- Topic of the *Birch and Swinnerton-Dyer conjecture*.
- Tool in Wiles' proof of *Fermat's last theorem*.
- Methods for primality testing and integer factorisation.
- Applications in *elliptic curve cryptography*.

Elliptic curves: formally

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Theorem

An elliptic curve over $\mathbb{Q}$ *is of the form*

$$E = \left\{(x, y) \in \mathbb{Q}^2 \mid y^2 = x^3 + Ax + B\right\} \cup \{\mathcal{O}\},$$

for some $A, B \in \mathbb{Q}$ *such that* $4A^3 + 27B^2 \neq 0$*, where* $\mathcal{O} = [0, 1, 0]$*.*

Weierstrass representations

$$a^3 + b^3 + c^3 - 5abc - 3\left(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2\right) = 0, \qquad a, b, c \in \mathbb{Z}$$

Proposition

The curve given by the equation is isomorphic to the following elliptic curves.

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The curve given by the equation is isomorphic to the following elliptic curves.

- $\left\{(x, y) \in \mathbb{Q}^2 \mid 6y^2 + 6xy + 6y = -91x^3 + 141x^2 + 15x - 1\right\} \cup \{\mathcal{O}\}.$

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- $\left\{(x, y) \in \mathbb{Q}^2 \mid y^2 = x^3 + \frac{1}{4}x^2 - \frac{467}{2}x + \frac{5409}{4}\right\} \cup \{\mathcal{O}\}$.

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- $\left\{(x,y) \in \mathbb{Q}^2 \mid y^2 = x^3 - 302643x + 63998478\right\} \cup \{\mathcal{O}\}.$

Weierstrass representations

$$a^3 + b^3 + c^3 - 5abc - 3\left(a^2 b + ab^2 + a^2 c + ac^2 + b^2 c + bc^2\right) = 0, \qquad a, b, c \in \mathbb{Z}$$

Proposition

The curve given by the equation is isomorphic to the following elliptic curves.

- $\{(x, y) \in \mathbb{Q}^2 \mid 6y^2 + 6xy + 6y = -91x^3 + 141x^2 + 15x - 1\} \cup \{\mathcal{O}\}$.
- $\{(x, y) \in \mathbb{Q}^2 \mid y^2 + xy - \frac{91}{6}y = x^3 + \frac{47}{2}x^2 - \frac{455}{12}x - \frac{8281}{216}\} \cup \{\mathcal{O}\}$.
- $\{(x, y) \in \mathbb{Q}^2 \mid y^2 + xy + y = x^3 - 234x + 1352\} \cup \{\mathcal{O}\}$.
- $\{(x, y) \in \mathbb{Q}^2 \mid y^2 = x^3 + \frac{1}{4}x^2 - \frac{467}{2}x + \frac{5409}{4}\} \cup \{\mathcal{O}\}$.
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Let $A = -302643$ and $B = 63998478$.

Weierstrass representations

$$a^3+b^3+c^3-5abc-3\left(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2\right)=0, \qquad a,b,c\in\mathbb{Z}$$

Proposition

The curve given by the equation is isomorphic to the following elliptic curves.

- $\{(x,y)\in\mathbb{Q}^2 \mid 6y^2+6xy+6y=-91x^3+141x^2+15x-1\}\cup\{\mathcal{O}\}$.
- $\{(x,y)\in\mathbb{Q}^2 \mid y^2+xy-\frac{91}{6}y=x^3+\frac{47}{2}x^2-\frac{455}{12}x-\frac{8281}{216}\}\cup\{\mathcal{O}\}$.
- $\{(x,y)\in\mathbb{Q}^2 \mid y^2+xy+y=x^3-234x+1352\}\cup\{\mathcal{O}\}$.
- $\{(x,y)\in\mathbb{Q}^2 \mid y^2=x^3+\frac{1}{4}x^2-\frac{467}{2}x+\frac{5409}{4}\}\cup\{\mathcal{O}\}$.
- $\{(x,y)\in\mathbb{Q}^2 \mid y^2=x^3+109x^2+224x\}\cup\{\mathcal{O}\}$.
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- $\{(x,y)\in\mathbb{Q}^2 \mid y^2=x^3-302643x+63998478\}\cup\{\mathcal{O}\}$.

Let $A=-302643$ and $B=63998478$. Overall invertible transformations:

$$\begin{cases} a=\frac{1}{72}x+\frac{1}{216}y-\frac{277}{24} \\ b=\frac{1}{72}x-\frac{1}{216}y-\frac{277}{24} \\ c=\frac{1}{6}x-\frac{95}{2} \end{cases} \qquad \begin{cases} x=\dfrac{1710a+1710b-831c}{6a+6b-c} \\ y=\dfrac{-9828a+9828b}{6a+6b-c} \end{cases}$$

A group law

$$E = \{(x, y) \in \mathbb{Q}^2 \mid y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}$$

Theorem

E is an abelian group $(E, +)$.

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$$-(x, y) = (x, -y).$$

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$$-(x, y) = (x, -y)\,.$$

- *The sum of two points is obtained by inverting the third point of intersection between the curve and the line joining the two points.*

$$P + Q = \begin{cases} S & P = (x, y)\,,\ Q = (x', y')\,,\ x \neq x' \\ R & P = Q = (x, y)\,,\ y \neq 0 \\ P & Q = \mathcal{O} \\ \mathcal{O} & P = Q = (x, 0) \end{cases} \,,$$

$$S = \left(\frac{\big(A + xx'\big)\big(x + x'\big) + 2\big(B - yy'\big)}{(x - x')^2}, \frac{\big(Ay' - x'^2 y\big)\big(3x + x'\big) + \big(x^2 y' - Ay\big)\big(x + 3x'\big) - 4B\big(y - y'\big)}{(x - x')^3} \right),$$

$$R = \left(\frac{x^4 - 2Ax^2 - 8Bx + A^2}{4y^2}, \frac{x^6 + 5Ax^4 + 20Bx^3 - 5A^2x^2 - 4ABx - A^3 - 8B^2}{8y^3} \right).$$

Proof of the group law

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Let C and D be projective algebraic curves over an algebraically closed field $\overline{K}$. Then C and D intersect at exactly $\deg C \deg D$ points counted with intersection multiplicity.

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Let C, D, E be projective algebraic cubic curves over an algebraically closed field $\overline{K}$ such that

$$C \cap E = \{P_1, \ldots, P_8, Q\}, \qquad D \cap E = \{P_1, \ldots, P_8, R\},$$

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$$a^3+b^3+c^3-5abc-3\left(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2\right)=0, \qquad a,b,c\in\mathbb{Z}.$$

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- *Apply the change of variables:*

$$\begin{cases} x = \dfrac{1710a+1710b-831c}{6a+6b-c} \\ y = \dfrac{-9828a+9828b}{6a+6b-c} \end{cases}.$$

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$$y^2 = x^3+Ax+B, \qquad (x,y)\in\mathbb{Q}^2.$$

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- *Terminate or repeat.*

Computation: failure

Choose an invalid solution:

$$(a, b, c) = (-1, 1, -1)\,.$$

Computation: failure

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$$(a, b, c) = (-1, 1, -1)\,.$$

- Apply the change of variables:

$$(x, y) = (831, 19656)\,.$$

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Choose an invalid solution:

$$(a, b, c) = (-1, 1, -1) .$$

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Compute multiples of point:

Computation: failure

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$$(a, b, c) = (-1, 1, -1)\,.$$

- Apply the change of variables:

$$(x, y) = (831, 19656)\,.$$

Compute multiples of point:

- $1\,(x, y) = (831, 19656)$.

Computation: failure

Choose an invalid solution:

$$(a, b, c) = (-1, 1, -1) .$$

- Apply the change of variables:

$$(x, y) = (831, 19656) .$$

Compute multiples of point:

- $1\,(x, y) = (831, 19656)$.
- $2\,(x, y) = (363, 1404)$.

Computation: failure

Choose an invalid solution:

$$(a, b, c) = (-1, 1, -1)\,.$$

- Apply the change of variables:

$$(x, y) = (831, 19656)\,.$$

Compute multiples of point:

- $1\,(x, y) = (831, 19656)$.
- $2\,(x, y) = (363, 1404)$.
- $3\,(x, y) = (327, 0)$.

Computation: failure

Choose an invalid solution:

$$(a, b, c) = (-1, 1, -1)\,.$$

- Apply the change of variables:

$$(x, y) = (831, 19656)\,.$$

Compute multiples of point:

- $1\,(x, y) = (831, 19656)$.
- $2\,(x, y) = (363, 1404)$.
- $3\,(x, y) = (327, 0)$.
- $4\,(x, y) = (363, -1404)$.

Computation: failure

Choose an invalid solution:

$$(a, b, c) = (-1, 1, -1) .$$

- Apply the change of variables:

$$(x, y) = (831, 19656) .$$

Compute multiples of point:

- $1\,(x, y) = (831, 19656)$.
- $2\,(x, y) = (363, 1404)$.
- $3\,(x, y) = (327, 0)$.
- $4\,(x, y) = (363, -1404)$.
- $5\,(x, y) = (831, -19656)$.

Computation: failure

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$$(a, b, c) = (-1, 1, -1)\,.$$

- Apply the change of variables:

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Compute multiples of point:

- $1\,(x, y) = (831, 19656)$.
- $2\,(x, y) = (363, 1404)$.
- $3\,(x, y) = (327, 0)$.
- $4\,(x, y) = (363, -1404)$.
- $5\,(x, y) = (831, -19656)$.
- $6\,(x, y) = \mathcal{O}$.

Computation: failure

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- $5\,(x, y) = (831, -19656)$.
- $6\,(x, y) = \mathcal{O}$.

This is a cyclic subgroup of order six.

Computation: success

Choose a trivial solution:

$$(a, b, c) = (11, 4, -1)\,.$$

Computation: success

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- Apply the change of variables:

$$(x, y) = (291, -756) \, .$$

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Compute multiples of point:

Computation: success

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Compute multiples of point:

- $1\,(x, y) = (291, -756)$.

Computation: success

Choose a trivial solution:

$$(a, b, c) = (11, 4, -1)\,.$$

- Apply the change of variables:

$$(x, y) = (291, -756)\,.$$

Compute multiples of point:

- $1\,(x, y) = (291, -756)$.
- $2\,(x, y) = \left(\frac{22107}{49}, -\frac{1506492}{343}\right)$.

Computation: success

Choose a trivial solution:

$$(a, b, c) = (11, 4, -1)\,.$$

- Apply the change of variables:

$$(x, y) = (291, -756)\,.$$

Compute multiples of point:

- $1\,(x, y) = (291, -756)$.
- $2\,(x, y) = \left(\frac{22107}{49}, -\frac{1506492}{343}\right)$.
  - Apply the change of variables:

$$(a, b, c) = (-8784, 5165, 9499)\,.$$

Computation: success

Choose a trivial solution:

$$(a, b, c) = (11, 4, -1)\,.$$

- Apply the change of variables:

$$(x, y) = (291, -756)\,.$$

Compute multiples of point:

- $1\,(x, y) = (291, -756)$.
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$$(a, b, c) = (-8784, 5165, 9499)\,.$$

- $3\,(x, y) = \left(-\frac{2694138}{11881}, -\frac{14243306490}{1295029}\right)$.

Computation: success

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- $3\,(x, y) = \left(-\frac{2694138}{11881}, -\frac{14243306490}{1295029}\right)$.
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$$(a, b, c) = (679733219, -375326521, 883659076)\,.$$

Computation: success

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$$(a, b, c) = (679733219, -375326521, 883659076)\,.$$

- $9\,(x, y) = \left(\frac{38233875800801600760636052090610526039633899163277191 42}{13514400292716288512070907945002943352692578000406921},\right.$

$$\left. -\frac{1587622549247318249299172296638373895912313166958011719500537215259315694916502670}{1571068668597978434556364707291896268838086945430031322196754390420280407346469}\right).$$

Computation: success

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  - Apply the change of variables:

$$(a, b, c) = (\textsc{apple}, \textsc{banana}, \textsc{pineapple})\,.$$

Further facts

The general equation is

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} = N, \qquad a, b, c \in \mathbb{N}^*, \qquad N \in \mathbb{Z}.$$

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- The curve is

$$E \cong \mathbb{Z}^r \oplus \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} & N = 2 \\ \mathbb{Z}/6\mathbb{Z} & \text{otherwise} \end{cases}, \qquad r \in \mathbb{N}^*.$$

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  - Proof by heights.

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- The smallest solution for $N=4$ is $(a,b,c)=(\textsc{apple},\textsc{banana},\textsc{pineapple})$.
  - Proof by heights.
- The smallest solution for $N=178$ has four hundred million digits.

Further facts

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$$E \cong \mathbb{Z}^r \oplus \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} & N = 2 \\ \mathbb{Z}/6\mathbb{Z} & \text{otherwise} \end{cases}, \qquad r \in \mathbb{N}^*.$$

  - The curve for $N = 4$ has $r = 1$.
- The smallest solution for $N = 4$ is $(a, b, c) = (\text{APPLE}, \text{BANANA}, \text{PINEAPPLE})$.
  - Proof by heights.
- The smallest solution for $N = 178$ has four hundred million digits.
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  - There are infinitely many even N with solutions.

Further references

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