Computing Dirichlet L-functions over global function fields

Young Researchers in Algebraic Number Theory

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Dirichlet characters and L-functions over $\mathbb{F}_p(t)$

A Dirichlet character of modulus $m \in \mathbb{Z}$ is a map $\chi_m : (\mathbb{Z}/m)^\times \to \mathbb{C}^\times$.

For a fixed ring of integers $\mathbb{F}_p[t]$ of $\mathbb{F}_p(t)$, a **Dirichlet character of modulus** $m \in \mathbb{F}_p[t]$ is a map $\chi_m : (\mathbb{F}_p[t]/m)^\times \to \mathbb{C}^\times$.

In both cases, their **(incomplete) Dirichlet L-function** is

$$L(\chi_m, s) := \prod_{v \nmid m} \frac{1}{1 - \chi_m(v) p_v^{-s \deg v}}.$$

Conjecture (Generalised extended Riemann hypothesis)

The non-trivial zeroes of $L(\chi_m, s)$ have real part equal to $\frac{1}{2}$.

Frustration: there are many implementations of Dirichlet characters and L-functions over number fields, but none over global function fields!

Structure of units over $\mathbb{Q}$

For a modulus m in either $R = \mathbb{Q}$ or $R = \mathbb{F}_p[t]$, writing $m = m_1^{e_1} \cdot \dots \cdot m_r^{e_r}$ as a product of prime powers gives an isomorphism of abelian groups

$$\mathrm{Hom}((R/m)^\times, \mathbb{C}^\times) \cong \prod_{k=1}^{r} \mathrm{Hom}((R/m_k^{e_k})^\times, \mathbb{C}^\times),$$

so it suffices to consider χ_{m^e} when $m \in R$ is prime.

Lemma

Let $m \in \mathbb{Z}$ be prime. Then

$$(\mathbb{Z}/m^e)^\times \cong \begin{cases} C_2 \times C_{2^{e-2}} & \text{if } m = 2 \text{ and } e \geq 3, \\ C_{m^{e-1}(m^e-1)} & \text{otherwise.} \end{cases}$$

Over $\mathbb{Q}$, Dirichlet characters are determined by its values on generators.

Structure of units over $\mathbb{F}_p(t)$

When $m \in \mathbb{F}_p[t]$ is prime, $(\mathbb{F}_p[t]/m^e)^\times$ is far from cyclic in general.

e	$(\mathbb{F}_2[t]/t^e)^\times$
1	C_1
2	C_2
3	C_4
4	$C_2 \times C_4$
5	$C_2 \times C_8$
6	$C_2^2 \times C_8$
7	$C_2 \times C_4 \times C_8$
8	$C_2^2 \times C_4 \times C_8$
9	$C_2^2 \times C_4 \times C_{16}$
10	$C_2^3 \times C_4 \times C_{16}$
11	$C_2^2 \times C_4^2 \times C_{16}$
12	$C_2^3 \times C_4^2 \times C_{16}$
13	$C_2^3 \times C_4 \times C_8 \times C_{16}$

e	$(\mathbb{F}_3[t]/t^e)^\times$
1	C_2
2	$C_2 \times C_3$
3	$C_2 \times C_3^2$
4	$C_2 \times C_3 \times C_9$
5	$C_2 \times C_3^2 \times C_9$
6	$C_2 \times C_3^3 \times C_9$
7	$C_2 \times C_3^2 \times C_9^2$
8	$C_2 \times C_3^3 \times C_9^2$
9	$C_2 \times C_3^4 \times C_9^2$
10	$C_2 \times C_3^4 \times C_9 \times C_{27}$
11	$C_2 \times C_3^5 \times C_9 \times C_{27}$
12	$C_2 \times C_3^6 \times C_9 \times C_{27}$
13	$C_2 \times C_3^5 \times C_9^2 \times C_{27}$

Question: where do these partitions come from?

Decomposition into canonical units

Lemma

Let $m \in \mathbb{F}_p[t]$ be prime of degree f, and let $h \in (\mathbb{F}_p[t]/m)^\times$ be fixed generators. Then for any $x \in (\mathbb{F}_p[t]/m^e)^\times$, there are unique exponents $1 \leq a \leq p^f - 1$ and $1 \leq b_{i,j} \leq p$ such that

$$x = h^a \cdot \prod_{i=1}^{e-1} \prod_{j=0}^{f-1} (1 + t^j m^i)^{b_{i,j}}.$$

Proof by algorithm.

Apply the division algorithm to give $y \equiv 1 \mod m$ and $z \in (\mathbb{F}_p[t]/m)^\times$ such that $x = y \cdot m + z$. Compute $a := \log_h \omega_p(z) \in \{1, \ldots, p^f - 1\}$, which is unique since $(\mathbb{F}_p[t]/m)^\times \cong C_{p^f - 1}$. Express y in base m:

$$y = 1 + (\textstyle\sum_{j=0}^{f-1} b_{1,j} t^j)m + (\sum_{j=0}^{f-1} b_{2,j} t^j)m^2 + \cdots + (\sum_{j=0}^{f-1} b_{e-1,j} t^j)m^{e-1}.$$

Replace y with $y \cdot \prod_{j=0}^{f-1}(1 + t^j m)^{-b_{1,j}} \equiv 1 \mod m^2$ and repeat. □

Dirichlet character example

Let $m := t^2 + 2 \in \mathbb{F}_5[t]$, and let $\chi_{m^4} : (\mathbb{F}_5[t]/m^4)^\times \to \mathbb{C}^\times$ be the (primitive) Dirichlet character given by

$$\begin{aligned} t+1 &\mapsto \zeta_{24}, \quad & 1+m &\mapsto \zeta_5, \quad & 1+m^2 &\mapsto \zeta_5^2, \quad & 1+m^3 &\mapsto \zeta_5^3, \\ & & 1+tm &\mapsto \zeta_5^4, \quad & 1+tm^2 &\mapsto \zeta_5^3, \quad & 1+tm^3 &\mapsto \zeta_5^2, \end{aligned}$$

noting that $(\mathbb{F}_5[t]/m^4)^\times \cong C_{24} \times C_5^6$. To evaluate $\chi_{m^4}(t^7+1)$, compute

$$\begin{aligned} t^7 + 1 &= (2t+1) + 2tm + 4tm^2 + tm^3 \\ &= (2t+1) \cdot (1 + (2+3t)m + 4tm^2 + (4+3t)m^3) \\ &= (2t+1) \cdot (1+m)^2(1+tm)^3 \cdot (1 + 3tm^2 + (1+t)m^3) \\ &= (2t+1) \cdot (1+m)^2(1+tm)^3 \cdot (1+tm^2)^3 \cdot (1 + (1+t)m^3) \\ &= (2t+1) \cdot (1+m)^2(1+tm)^3 \cdot (1+tm^2)^3 \cdot (1+m^3)(1+tm^3). \end{aligned}$$

Then $2t + 1 \equiv (t+1)^{22} \mod m$, so

$$\chi_{m^4}(t^7+1) = \zeta_{24}^{22} \cdot \zeta_5^2(\zeta_5^4)^3 \cdot (\zeta_5^3)^3 \cdot \zeta_5^3\zeta_5^2 = \zeta_{60}^{31}.$$

Dirichlet characters over $\mathbb{F}_q(C)$

In general, a global function field is the function field $\mathbb{F}_q(C)$ of a smooth proper geometrically irreducible curve C of genus g over a finite field $\mathbb{F}_q$.

A (primitive) Dirichlet character over $\mathbb{F}_q(C)$ of modulus $m \subseteq \mathcal{O}_v$ really should be a complex character of the ~~ray class group modulo m (Weber) idèle class group I trivial on $1 + m$ (Hecke)~~ absolute Galois group $G := \mathrm{Gal}(\overline{\mathbb{F}_q(C)}/\mathbb{F}_q(C))$ that factors through a finite abelian extension of $\mathbb{F}_q(C)$ defined with the Drinfeld module associated to m (Artin).

In particular, Artin reciprocity gives a map $I \to G$ that sends a place v of $\mathbb{F}_q(C)$ to (a choice of) a geometric Frobenius Fr_v^{-1} in G.

For a Dirichlet character $\chi_m : G \to \mathbb{C}^\times$, denote

$$\chi_m(v) := \begin{cases} \chi_m(\mathrm{Fr}_v^{-1}) & \text{if } v \text{ is unramified,} \\ 0 & \text{if } v \text{ is ramified.} \end{cases}$$

Artin conductors over $\mathbb{F}_q(C)$

The **Artin conductor** of $\chi_m : G \to \mathbb{C}^\times$ is the effective Weil divisor

$$\mathfrak{f}(\chi_m) := \sum_v \alpha_v(\chi_m)[v], \qquad \alpha_v(\chi_m) := \sum_{\chi_m(G_{v,i}) \neq 0} \frac{1}{[G_{v,0} : G_{v,i}]} \in \mathbb{N}.$$

where v runs over all of the closed points of C.

When $C = \mathbb{P}^1_{\mathbb{F}_q}$, after fixing a place at infinity ∞,

$$\{\text{closed points of } C\} \quad \leftrightsquigarrow \quad \{\text{primes of } \mathbb{F}_q[t]\} \cup \{\infty\}.$$

In fact, it turns out that

$$\alpha_v(\chi_m) = \begin{cases} v(m) & \text{if } v \in \mathbb{F}_q[t], \\ 1 & \text{if } v = \infty \text{ and } \chi_m|_{\mathbb{F}_q^\times} \not\equiv 1, \\ 0 & \text{if } v = \infty \text{ and } \chi_m|_{\mathbb{F}_q^\times} \equiv 1, \end{cases}$$

and in the final case $\chi_m(\infty) = 1$.

Dirichlet L-functions over $\mathbb{F}_q(C)$

The **formal L-function** of $\chi_m : G \to \mathbb{C}^\times$ is the power series

$$\mathcal{L}(\chi_m, T) := \prod_v (1 - \chi_m(v) T^{\deg v})^{-1} \in \mathbb{C}[[T]],$$

and $L(\chi_m, s) := \mathcal{L}(\chi_m, q^{-s})$ is its **(complete) Dirichlet L-function**.

If $\{c_{v,n}\}_{n=0}^{\infty}$ are the coefficients of $(1 - \chi_m(v) T^{\deg v})^{-1}$, then

$$\begin{aligned}\mathcal{L}(\chi_m, T) &= \prod_v \left(\sum_{n=0}^{\infty} c_{v,n} T^{n \deg v} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{\deg D = n} c_D \right) T^n,\end{aligned}$$

where $c_D := \prod_v c_{v,n_v}$ for any effective Weil divisor $D = \sum_v n_v[v]$ on C.

Rationality and the functional equation

On the other hand, $\mathcal{L}(\chi_m, T)$ is essentially the ζ-function of C.

Corollary (of the Weil conjectures)

Let $\chi_m : G \to \mathbb{C}^\times$ be a Dirichlet character over $\mathbb{F}_q(C)$ that is ramified somewhere. Then $\mathcal{L}(\chi_m, T)$ is a polynomial of degree

$$d(\chi_m) := 2g - 2 + \deg \mathfrak{f}(\chi_m).$$

Furthermore, $\mathcal{L}(\chi_m, T)$ satisfies the functional equation

$$\mathcal{L}(\chi_m, T) = \epsilon(\chi_m) \cdot (\sqrt{q}T)^{d(\chi_m)} \cdot \overline{\mathcal{L}(\chi_m, (qT)^{-1})},$$

for some root number $\epsilon(\chi_m) \in \mathbb{C}^\times$ defined with Gauss sums.

The fact that $\deg \mathcal{L}(\chi_m, T) = d(\chi_m)$ means that it is determined by its coefficients c_D for all effective Weil divisors D on C with $\deg D \leq d(\chi_m)$.

Dirichlet L-function example with rationality

Let $m := t^3 + 2t + 1 \in \mathbb{F}_3[t]$, and let $\chi_m : (\mathbb{F}_3[t]/m)^\times \to \mathbb{C}^\times$ be the (primitive) Dirichlet character given by $t \mapsto \zeta := \zeta_{26}$. Then

$$\deg \mathcal{L}(\chi_m, T) = d(\chi_m) = 2(0) - 2 + \deg([m] + [\infty]) = 2.$$

v	$1 - \chi_m(v)T$	$1 - \chi_m(v)T^{\deg v}$	$(1 - \chi_m(v)T^{\deg v})^{-1}$
∞	1	1	1
t	$1 - \zeta T$	$1 - \zeta T$	$1 + \zeta T + \zeta^2 T^2 + \dots$
$t+1$	$1 - \zeta^9 T$	$1 - \zeta^9 T$	$1 + \zeta^9 T + \zeta^{18} T^2 + \dots$
$t+2$	$1 - \zeta^3 T$	$1 - \zeta^3 T$	$1 + \zeta^3 T + \zeta^6 T^2 + \dots$
t^2+1	$1 - \zeta^{21} T$	$1 - \zeta^{21} T^2$	$1 + \zeta^{21} T^2 + \dots$
t^2+t+2	$1 - \zeta^{11} T$	$1 - \zeta^{11} T^2$	$1 + \zeta^{11} T^2 + \dots$
t^2+2t+2	$1 - \zeta^7 T$	$1 - \zeta^7 T^2$	$1 + \zeta^7 T^2 + \dots$

The product of $(1 - \chi_m(v)T^{\deg v})^{-1}$ computes to be

$$1 + (\zeta^9 + \zeta^3 + \zeta)T + (2\zeta^{11} + \zeta^9 - 2\zeta^8 + 2\zeta^7 + \zeta^3 + \zeta - 1)T^2 + \dots.$$

Thus $\mathcal{L}(\chi_m, T)$ is just the first three terms!

Application of the functional equation

The functional equation $\mathcal{L}(\chi_m, T) = \epsilon(\chi_m) \cdot (\sqrt{q}T)^{d(\chi_m)} \cdot \overline{\mathcal{L}(\chi_m, (qT)^{-1})}$ reduces the required computation by $\lfloor d(\chi_m)/2 \rfloor$.

If $\{c_n\}_{n=0}^{d(\chi_m)}$ are the coefficients of $\mathcal{L}(\chi_m, T)$, then this says

$$\begin{aligned}
\sum_{n=0}^{d(\chi_m)} (c_n \cdot T^n) &= \sum_{n=0}^{d(\chi_m)} \left(\epsilon(\chi_m) \cdot \sqrt{q}^{d(\chi_m)-2n} \cdot \overline{c_n} \cdot T^{d(\chi_m)-n}\right) \\
&= \sum_{n=0}^{d(\chi_m)} \left(\epsilon(\chi_m) \cdot \sqrt{q}^{2n-d(\chi_m)} \cdot \overline{c_{d(\chi_m)-n}} \cdot T^n\right).
\end{aligned}$$

In other words, when $\lceil d(\chi_m)/2 \rceil \leq n \leq d(\chi_m)$,

$$c_n = \epsilon(\chi_m) \cdot \sqrt{q}^{2n-d(\chi_m)} \cdot \overline{c_{d(\chi_m)-n}},$$

so $\mathcal{L}(\chi_m, T)$ is determined by its coefficients c_D for all effective Weil divisors D on C with $\deg D \leq \lfloor d(\chi_m)/2 \rfloor$ once $\epsilon(\chi_m)$ is computed.

Dirichlet L-function example with functional equation

Let $m := t^3 + 2t + 1 \in \mathbb{F}_3[t]$, and let $\chi_{m^2} : (\mathbb{F}_3[t]/m^2)^\times \to \mathbb{C}^\times$ be the (primitive) Dirichlet character given by

$$t \mapsto \zeta_{13}, \qquad 1 + m \mapsto \zeta_3, \qquad 1 + tm \mapsto \zeta_3^2, \qquad 1 + t^2 m \mapsto \zeta_3,$$

noting that $(\mathbb{F}_3[t]/m^2)^\times \cong C_{26} \times C_3^3$ and $\chi_{m^2}(2) = 1$. Then

$$\deg \mathcal{L}(\chi_{m^2}, T) = d(\chi_{m^2}) = 2(0) - 2 + \deg(2[m]) = 4.$$

By a similar computation as before,

$$\mathcal{L}(\chi_{m^2}, T) \equiv 1 + ZT - (Z+1)T^2 \mod T^3,$$

where $Z := \zeta_{13}^9 + \zeta_{13}^3 + \zeta_{13}$. This forces $Z + 1 = \epsilon(\chi_m) \cdot \overline{(Z+1)}$! Thus

$$\mathcal{L}(\chi_{m^2}, T) = 1 + ZT - (Z+1)T^2 + 3\epsilon(\chi_m)\overline{Z}T^3 + 9\epsilon(\chi_m)T^4.$$

Alternatively, $\epsilon(\chi_m)$ can be computed manually, in which case it suffices to determine the first two terms of $\mathcal{L}(\chi_{m^2}, T)$.

Motivic L-functions over $\mathbb{F}_q(C)$

In general, the formal L-function of an almost everywhere unramified ℓ-adic representation $\rho : G \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ over $\mathbb{F}_q(C)$ is given by

$$\mathcal{L}(\rho, T) := \prod_v \det(1 - \rho^{I_v}(v) T^{\deg v})^{-1} \in \overline{\mathbb{Q}_\ell}[[T]].$$

Corollary (of the proof of the Weil conjectures)

Let $\rho : G \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ be an ℓ-adic representation over $\mathbb{F}_q(C)$ that is ramified somewhere. Then $\mathcal{L}(\rho, T)$ is a polynomial of degree

$$d(\rho) := (2g - 2) \dim \rho + \deg \mathfrak{f}(\rho).$$

Furthermore, $\mathcal{L}(\rho, T)$ satisfies the functional equation

$$\mathcal{L}(\rho, T) = \epsilon(\rho) \cdot (q^{(w(\rho)+1)/2} T)^{d(\rho)} \cdot \mathcal{L}(\rho, 1/q^{w(\rho)+1} T)^{g(\rho)},$$

where $w(\rho)$ is the weight of ρ and $g(\rho)$ is some automorphism on $\overline{\mathbb{Q}_\ell}$.

Concluding remarks

I have implemented Magma intrinsics for computing formal L-functions of general ℓ-adic representations over $\mathbb{F}_q(C)$, including specific examples:

- Dirichlet characters with semi-efficient root numbers
- elliptic curves with efficient root numbers except when $q = 2, 3$, which is faster than existing functionality when $q = 2, 3, 5, 7$
- tensor products with coprime conductors

Theorem

Let $\rho, \sigma : G \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ *be* ℓ*-adic representations of* $\mathbb{F}_q(C)$ *with coprime Artin conductors. Then* $\deg \mathfrak{f}(\rho \otimes \sigma) = \deg \mathfrak{f}(\rho) \dim \sigma + \deg \mathfrak{f}(\sigma) \dim \rho$ *and*

$$\epsilon(\rho \otimes \sigma) = \epsilon(\rho)^{\dim \sigma} \cdot \epsilon(\sigma)^{\dim \rho} \cdot \frac{\det \sigma(\mathfrak{f}(\rho))}{|\det \sigma(\mathfrak{f}(\rho))|} \cdot \frac{\det \rho(\mathfrak{f}(\sigma))}{|\det \rho(\mathfrak{f}(\sigma))|}.$$

I believe that having a systematic method to compute formal L-functions will be useful in creating databases of motives over global function fields!