Computing Euler factors

Study group on models of curves and arithmetic applications

David Kurniadi Angdinata

London School of Geometry and Number Theory

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Notation

p an odd prime (of almost good reduction)

C a smooth projective (hyperelliptic) curve of genus g $(= 2)$ over $\mathbb{Q}$ (with semistable reduction at p) given by an integral model

$$Y^2 = c \prod_{r \in \mathcal{R}} (X - r), \qquad c \in \mathbb{Z}, \qquad r \in \overline{\mathbb{Q}}.$$

$\mathcal{C}$ the minimal regular model of C at p

$\widetilde{\mathcal{C}}$ the special fibre of $\mathcal{C}$

$\overline{C}$ the base change of C to $\overline{\mathbb{Q}}$

$\overline{\widetilde{\mathcal{C}}}$ the base change of $\widetilde{\mathcal{C}}$ to $\overline{\mathbb{F}_p}$

L-functions

Recall that the L-function of C is the Euler product

$$L(C,s) := \prod_p \frac{1}{L_p(C,p^{-s})},$$

over all primes p, where the local Euler factor at p is the polynomial

$$L_p(C,T) := \det(1 - T \cdot \mathrm{Frob}_p^{-1} \mid H^1_{\text{ét}}(\overline{C}, \mathbb{Q}_\ell)^{I_p}).$$

When C has semistable reduction at p,

$$H^1_{\text{ét}}(\overline{C}, \mathbb{Q}_\ell)^{I_p} \cong H^1_{\text{ét}}(\overline{\widetilde{\mathcal{C}}}, \mathbb{Q}_\ell),$$

which is an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-representations, so that

$$L_p(C,T) = \det(1 - T \cdot \mathrm{Frob}_p^{-1} \mid H^1_{\text{ét}}(\overline{\widetilde{\mathcal{C}}}, \mathbb{Q}_\ell)).$$

This can be extracted from the ζ-function of $\widetilde{\mathcal{C}}$.

ζ-functions

The ζ-function of a projective curve X over $\mathbb{F}_p$ is the rational function

$$\zeta(X, T) := \exp\left(\sum_{k\geq 1} \#X(\mathbb{F}_{p^k})\frac{T^k}{k}\right) = \frac{P_1(X, T)}{P_0(X, T)\cdot P_2(X, T)},$$

by the Weil conjectures, where

$$P_i(X, T) := \det(1 - T\cdot \mathrm{Frob}_p^{-1} \mid H^1_{\text{ét}}(X, \mathbb{Q}_\ell)), \qquad i = 0, 1, 2.$$

When the Jacobian $\mathrm{Jac}(C)$ of C has good reduction at p,

$$P_0(\widetilde{\mathcal{C}}, T) = 1 - T, \qquad \deg P_1(\widetilde{\mathcal{C}}, T) = 2g, \qquad P_2(\widetilde{\mathcal{C}}, T) = 1 - pT,$$

so that $L_p(C, T)$ is determined by $\#\widetilde{\mathcal{C}}(\mathbb{F}_{p^k})$ for sufficiently many $k \geq 1$.

In general, this requires computing the minimal regular model $\mathcal{C}$ by a resolution of singularities, which is computationally expensive.

Cluster pictures

Instead of computing $\mathcal{C}$, its special fibre $\widetilde{\mathcal{C}}$ can be recovered from cluster picture machinery, with explicit models for its irreducible components.

Recall that a cluster is a non-empty subset of $\mathcal{R}$ of the form

$$\mathfrak{s} = \{r \in \mathcal{R} \mid \nu_p(r - z) \geq d\}, \qquad z \in \overline{\mathbb{Q}_p}, \qquad d \in \mathbb{Q}.$$

The depth $d_\mathfrak{s}$ of a cluster $\mathfrak{s}$ is the largest such d, in which case any z with $\nu_p(r - z) = d_\mathfrak{s}$ for some $r \in \mathfrak{s}$ is called a centre $z_\mathfrak{s}$ of $\mathfrak{s}$. A child of $\mathfrak{s}$ is a maximal subcluster $\mathfrak{s}' \subsetneq \mathfrak{s}$, and its relative depth $\delta_{\mathfrak{s}'}$ is simply $d_{\mathfrak{s}'} - d_\mathfrak{s}$.

A cluster $\mathfrak{s}$ is called odd or even if $|\mathfrak{s}|$ is odd or even respectively. It is called übereven if every child of $\mathfrak{s}$ is even. It is called twin if $|\mathfrak{s}| = 2$, and it is called cotwin if it is not übereven but it has a child $\mathfrak{s}'$ with $|\mathfrak{s}'| = 2g$.

The cluster $\mathcal{R}$ is called principal if it is odd or if it has more than two children. In general, a cluster $\mathfrak{s}$ is called principal when $|\mathfrak{s}| \geq 3$ but has no children $\mathfrak{s}'$ with $|\mathfrak{s}'| = 2g$.

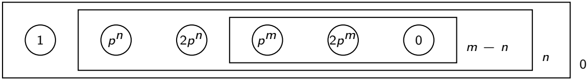
Cluster picture example

Let C be the hyperelliptic curve over $\mathbb{Q}$ given by

$$Y^2 = 2X(X-1)(X-p^n)(X-2p^n)(X-p^m)(X-2p^m),$$

where p is an odd prime and $m \geq n$ are positive integers.

The associated cluster picture is:



The cluster $\mathcal{R}$ is not principal, but it has two principal subclusters.

- The odd subcluster $\mathfrak{s}_m := \{p^m, 2p^m, 0\}$ has centre $z_{\mathfrak{s}_m} = 0$, depth $d_{\mathfrak{s}_m} = m$ and relative depth $\delta_{\mathfrak{s}_m} = m - n$.
- The odd subcluster $\mathfrak{s}_n := \{p^n, 2p^n, p^m, 2p^m, 0\}$ has centre $z_{\mathfrak{s}_n} = p^n$, depth $d_{\mathfrak{s}_n} = n$, and relative depth $\delta_{\mathfrak{s}_n} = n$.

Neither subclusters are übereven or cotwin.

Components of special fibres

Theorem (M2D2, Theorem 8.6(1))

Let C be a hyperelliptic curve over $\mathbb{Q}$ given by

$$Y^2 = c\prod_{r\in\mathcal{R}}(X-r), \qquad c\in\mathbb{Z}, \qquad r\in\overline{\mathbb{Q}}.$$

Assume that C has semistable reduction at some odd prime p, and that $\delta_{\mathfrak{s}} \neq \frac{1}{2}$ for any principal cluster $\mathfrak{s}$. Then the components of $\widetilde{\mathcal{C}}$ consist of the curves $\Gamma_{\mathfrak{s}}$ associated to principal clusters $\mathfrak{s}$, given by

$$Y^2 = \widetilde{\frac{c}{p^{\nu_p(c)}}} \prod_{r\in\mathcal{R}\setminus\mathfrak{s}} \widetilde{\frac{z_{\mathfrak{s}}-r}{p^{\nu_p(z_{\mathfrak{s}}-r)}}} \prod_{\text{odd } \mathfrak{s}'<\mathfrak{s}} \left(X - \widetilde{\frac{z_{\mathfrak{s}'}-z_{\mathfrak{s}}}{p^{d_{\mathfrak{s}}}}}\right).$$

This is irreducible except when $\mathfrak{s}$ is übereven, in which case it has two irreducible components $\Gamma_{\mathfrak{s}}^{+}$ and $\Gamma_{\mathfrak{s}}^{-}$. The remaining components of $\widetilde{\mathcal{C}}$ are chains of $\mathbb{P}^1$ that link $\Gamma_{\mathfrak{s}}$, which are given by the following conditions.

Intersections of special fibres

Theorem (M2D2, Theorem 8.6(1), continued)

- *Assume that* $\mathfrak{s}' < \mathfrak{s}$.
  - *When* $\mathfrak{s}'$ *is principal odd and* $\mathfrak{s}$ *is principal, then there is a chain from* $\Gamma_{\mathfrak{s}'}$ *to* $\Gamma_{\mathfrak{s}}$ *of length* $\frac{1}{2}\delta_{\mathfrak{s}'} - 1$.
  - *When* $\mathfrak{s}'$ *is principal even and* $\mathfrak{s}$ *is principal, then there are two chains from* $\Gamma^+_{\mathfrak{s}'}$ *to* $\Gamma^+_{\mathfrak{s}}$ *and from* $\Gamma^-_{\mathfrak{s}'}$ *to* $\Gamma^-_{\mathfrak{s}}$ *each of length* $\delta_{\mathfrak{s}'} - 1$.
  - *When* $\mathfrak{s}'$ *is twin and* $\mathfrak{s}$ *is principal, then there is a chain from* $\Gamma^-_{\mathfrak{s}}$ *to* $\Gamma^+_{\mathfrak{s}}$ *of length* $2\delta_{\mathfrak{s}'} - 1$.
  - *When* $\mathfrak{s}'$ *is principal and* $\mathfrak{s}$ *is cotwin, then there is a chain from* $\Gamma^-_{\mathfrak{s}'}$ *to* $\Gamma^+_{\mathfrak{s}'}$ *of length* $2\delta_{\mathfrak{s}'} - 1$.
- *Assume that* $\mathcal{R}$ *is not principal, but* $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$.
  - *When* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are principal odd, then there is a chain from* $\Gamma_{\mathfrak{s}_1}$ *to* $\Gamma_{\mathfrak{s}_2}$ *of length* $\frac{1}{2}(\delta_{\mathfrak{s}_1} + \delta_{\mathfrak{s}_2}) - 1$.
  - *When* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are principal even, then there are two chains from* $\Gamma^+_{\mathfrak{s}_1}$ *to* $\Gamma^+_{\mathfrak{s}_2}$ *and from* $\Gamma^-_{\mathfrak{s}_1}$ *to* $\Gamma^-_{\mathfrak{s}_2}$ *each of length* $\delta_{\mathfrak{s}_1} + \delta_{\mathfrak{s}_2} - 1$.
  - *When* $\mathfrak{s}_1$ *is principal even and* $\mathfrak{s}_2$ *is twin, then there is a chain from* $\Gamma^-_{\mathfrak{s}_1}$ *to* $\Gamma^+_{\mathfrak{s}_1}$ *of length* $2(\delta_{\mathfrak{s}_1} + \delta_{\mathfrak{s}_2}) - 1$.

Special fibre example

Continuing on the previous example, $\Gamma_{\mathfrak{s}_m}$ computes to be

$$
\begin{aligned}
Y^2 = 2&\left(\frac{\widetilde{0-1}}{p^{\nu_p(0-1)}}\right)\left(\frac{\widetilde{0-p^n}}{p^{\nu_p(0-p^n)}}\right)\left(\frac{\widetilde{0-2p^n}}{p^{\nu_p(0-2p^n)}}\right)\\
&\left(X-\frac{\widetilde{p^m-0}}{p^m}\right)\left(X-\frac{\widetilde{2p^m-0}}{p^m}\right)\left(X-\frac{\widetilde{0-0}}{p^m}\right)\\
= -4X(X-1)(X-2),
\end{aligned}
$$

and $\Gamma_{\mathfrak{s}_n}$ computes to be

$$
\begin{aligned}
Y^2 &= 2\left(\frac{\widetilde{p^n-1}}{p^{\nu_p(p^n-1)}}\right)\left(X-\frac{\widetilde{p^n-p^n}}{p^n}\right)\left(X-\frac{\widetilde{2p^n-p^n}}{p^n}\right)\left(X-\frac{\widetilde{0-p^n}}{p^n}\right)\\
&= -2X(X-1)(X+1).
\end{aligned}
$$

Furthermore, there is a chain from $\Gamma_{\mathfrak{s}_m}$ to $\Gamma_{\mathfrak{s}_n}$ of length $\frac{1}{2}(m-n)-1$.

ζ-function example

Recall that to compute $\zeta(\widetilde{\mathcal{C}}, T)$, it suffices to compute

$$\#C(\mathbb{F}_{p^k}) = \#\Gamma_{\mathfrak{s}_m}(\mathbb{F}_{p^k}) + \#\Gamma_{\mathfrak{s}_n}(\mathbb{F}_{p^k}) + \left(\frac{m-n}{2} - 1\right)\#\mathbb{P}^1(\mathbb{F}_{p^k}) - \frac{m-n}{2},$$

for all $k \geq 1$. For instance, if $p = 5$, then

$$\begin{aligned}
\#\Gamma_{\mathfrak{s}_m}(\mathbb{F}_{5^k}) &= 1 - ((-1-2i)^k + (-1+2i)^k) + 5^k, \\
\#\Gamma_{\mathfrak{s}_n}(\mathbb{F}_{5^k}) &= 1 - ((1-2i)^k + (1+2i)^k) + 5^k,
\end{aligned}$$

and if $m = 16$ and $n = 10$, then

$$\#C(\mathbb{F}_{5^k}) = 1 + 4 \cdot 5^k - \sum_{\pm}(\pm 1 \pm 2i)^k,$$

so that

$$\zeta(\widetilde{\mathcal{C}}, T) = \frac{\prod_{\pm}(1 - (\pm 1 \pm 2i)T)}{(1-T)(1-5T)^4} = \frac{(1-2T+5T^2)(1+2T+5T^2)}{(1-T)(1-5T)^4}.$$

Almost good primes

Unlike elliptic curves, there are higher genus curves C over $\mathbb{Q}$ with primes p that divide its minimal discriminant Δ_C but do not divide its conductor $\mathfrak{f}_C$, such as when $\mathrm{Jac}(C)$ reduces to a product of elliptic curves over $\mathbb{F}_p$. These primes p are called primes of **almost good reduction** for C.

For instance, the genus two curve over $\mathbb{Q}$ given by

$$\begin{aligned} Y^2 + (X^3 + X^2 + X)Y = {} & -144061786290072X^6 - 23062462482396X^5 \\ & - 1266273619292236X^4 - 3052943051575761X^3 \\ & + 3989955132045666X^2 + 3438312415pX - 1707513566p \end{aligned}$$

has $\mathfrak{f}_C = 270761$ but $\Delta_C = 270761p^{22}$ where $p = 14556001$.

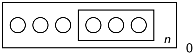
Maistret and Sutherland were motivated to expand the LMFDB, which currently contains 66158 genus two curves C over $\mathbb{Q}$ with $\Delta_C \leq 10^6$, to over $5 \cdot 10^6$ genus two curves C over $\mathbb{Q}$ with $\mathfrak{f}_C \leq 2^{20} \approx 10^6$.

Cluster pictures at almost good primes

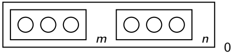
The prime of almost good reduction forces the existence of a subcluster of size 3, all subclusters to be odd, and specific conditions on their depths.

Theorem (MS25, Corollaries 3.5/7/10/11)

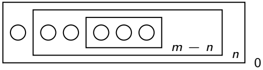
Let C be a hyperelliptic curve over $\mathbb{Q}$ given by $Y^2 = \sum_{i=0}^{6} c_i X^i \in \mathbb{Z}[X]$ such that $d_{\mathcal{R}} = 0$ and $\nu_p(c_6) = \min_i \nu_p(c_i) \leq 1$ at some odd prime p of almost good reduction. Then its cluster picture is one of the following:



where $\nu_p(c_6) = 0 \equiv n \mod 2$



where $m \geq n$ *and* $\nu_p(c_6) \equiv m \equiv n \mod 2$



where $m > n$ *and* $\nu_p(c_6) \equiv m \equiv n \mod 2$

Furthermore, there is an explicit description for $\widetilde{\mathcal{C}}$ as the union of two elliptic curves over $\mathbb{F}_{p^2}$ linked by a chain of $\mathbb{P}^1$ for each cluster picture.

Computing Euler factors

It turns out that any genus two curve over $\mathbb{Q}$ with almost good reduction at an odd prime can be normalised to obtain such a model.

Theorem (MS25, Theorem 1.1)

Let C be a genus two curve over $\mathbb{Q}$ given by $Y^2 = \sum_{i=0}^{6} c_i X^i \in \mathbb{Z}[X]$ with almost good reduction at some odd prime p. Then there is a probabilistic algorithm that computes $L_p(C, T)$ with running time

$$O\left(\frac{(\max_i \log|c_i|)^2 \log^2(\max_i \log|c_i|)}{\log p} + \log^5 p\right).$$

Furthermore, if a quadratic non-residue modulo p is given, then the algorithm is deterministic with the same running time.

This has been implemented in Magma in the public `Genus2Euler` repository. In a test on 3454506 pairs of (C, p), it is almost 5000 times faster than the existing `EulerFactor` function in Magma, including 489 pairs of (C, p) whose computations were terminated after eight hours.

References

MS25 Céline Maistret, Andrew Sutherland. Computing Euler factors of genus 2 curves at odd primes of almost good reduction. Research in Number Theory 11, 37 (2025). https://doi.org/10.1007/s40993-024-00605-7

M2D2 Tim Dokchitser, Vladimir Dokchitser, Céline Maistret, Adam Morgan. Arithmetic of hyperelliptic curves over local fields. Mathematische Annalen 385, 1213-1322 (2023). https://doi.org/10.1007/s00208-021-02319-y