Computing L-functions over global function fields

Elliptic Curves in the Cotswolds

David Kurniadi Angdinata

London School of Geometry and Number Theory

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Global fields

Let E be an elliptic curve over a global field K. Its L-function is given by

$$L(E,s) := \prod_v \frac{1}{\mathcal{L}_v(E, p_v^{-s \deg v})},$$

where p_v is the residue characteristic at each place v of K.

Here, the local Euler factors are given by

$$\mathcal{L}_v(E,T) := \det(1 - T \cdot \phi_v^{-1} \mid \rho_{E,\ell}^{I_v}) \in 1 + T \cdot \mathbb{Q}[T],$$

where ℓ is some prime different from p_v.

Conjecture (Birch and Swinnerton-Dyer)

The arithmetic of E is determined by the analysis of $L(E,s)$ at $s=1$.

There is much numerical evidence, which requires computing $L(E,s)$!

Computing special values

Over a number field K, Dokchitser [1] gave an algorithm to compute the special values of $L(E,s)$ assuming the functional equation

$$\Lambda(E,s) = \epsilon_E \,\mathrm{Nm}(\mathfrak{f}_E)^{1-s}\Delta_K^{1-s}\Lambda(E,2-s),$$

where its completed L-function is given by

$$\Lambda(E,s) := \left(\frac{\Gamma(s)}{(2\pi)^s}\right)^{[K:\mathbb{Q}]} L(E,s).$$

This was originally the `ComputeL` package in PARI/GP, but later ported to Magma as `LSeries()` and SageMath as `lseries().dokchitser()`.

Over a global function field, Magma has `LFunction()`, which uses the theory of Mordell–Weil lattices on elliptic surfaces to give a polynomial.

I claim that there is a much easier way to compute the same polynomial!

[1] Tim Dokchitser. "Computing special values of motivic L-functions" Experimental Mathematics 13 (2) 137–150, 2004

Global function fields

Let $K := k(C)$ be the global function field of a smooth proper geometrically irreducible curve C over a finite field $k := \mathbb{F}_q$.

The formal L-function of an elliptic curve E over K is given by

$$\mathcal{L}(E, T) := \prod_v \frac{1}{\mathcal{L}_v(E, T^{\deg v})} \in \mathbb{Q}[[T]],$$

so that $L(E, s) = \mathcal{L}(E, q^{-s})$.

If $\{a_{v,i}\}_{i=0}^{\infty}$ are the coefficients of $\mathcal{L}_v(E, T^{\deg v})^{-1}$, then

$$\mathcal{L}(E, T) = \prod_v \left(\sum_{i=0}^{\infty} a_{v,i} T^{i \deg v} \right) = \sum_{j=0}^{\infty} \left(\sum_{\deg D = j} a_D \right) T^j,$$

where $a_D := \prod_v a_{v,i_v}$ for any effective Weil divisor $D = \sum_v i_v [v]$ on C.

Rationality

Corollary (of the Weil conjectures [2])

There are polynomials $P_0(T), P_1(T), P_2(T) \in 1 + T \cdot \mathbb{Q}[T]$ *such that*

$$\mathcal{L}(E,T) = \frac{P_1(T)}{P_0(T) \cdot P_2(T)} \in \mathbb{Q}(T),$$

and

$$-\deg P_0(T) + \deg P_1(T) - \deg P_2(T) = 4g_C - 4 + \deg \mathfrak{f}_E.$$

Furthermore, there are simple expressions for $P_0(T)$ *and* $P_2(T)$ *in terms of* $\mathcal{L}(C,T)$*, and in fact* $P_0(T) = P_2(T) = 1$ *whenever* E *is not constant.*

Thus $\mathcal{L}(E,T)$ is completely determined by the coefficients a_D for all effective Weil divisors D on C with $\deg D \leq d_E$, where

$$d_E := 4g_C - 4 + \deg \mathfrak{f}_E + \deg P_0(T) + \deg P_2(T).$$

[2] Grothendieck–Lefschetz trace formula and Grothendieck–Ogg–Shafarevich formula

Quadratic example

Let E be the elliptic curve $y^2 = x^3 + x^2 + t^2 + 2$ over $K = \mathbb{F}_3(t)$. Then

$$\deg \mathcal{L}(E, T) = d_E = 4(0) - 4 + \deg(4[\tfrac{1}{t}] + [t+1] + [t+2]) = 2.$$

v	$\mathcal{L}_v(E,T)$	$\mathcal{L}_v(E, T^{\deg v})$	$\mathcal{L}_v(E, T^{\deg v})^{-1}$
$\frac{1}{t}$	1	1	1
t	$1 - T + 3T^2$	$1 - T + 3T^2$	$1 + T - 2T^2 + \dots$
$t+1$	$1 - T$	$1 - T$	$1 + T + T^2 + \dots$
$t+2$	$1 - T$	$1 - T$	$1 + T + T^2 + \dots$
t^2+1	$1 + 2T + 3T^2$	$1 + 2T^2 + \dots$	$1 - 2T^2 + \dots$
t^2+t+2	$1 - 4T + 3T^2$	$1 - 4T^2 + \dots$	$1 + 4T^2 + \dots$
t^2+2t+2	$1 - 4T + 3T^2$	$1 - 4T^2 + \dots$	$1 + 4T^2 + \dots$

Thus

$$\begin{aligned} \mathcal{L}(E,T) &\equiv (1 + T - 2T^2 + \dots) \cdot \dots \cdot (1 + 4T^2 + \dots) \mod T^3 \\ &\equiv 1 + 3T + 9T^2 \mod T^3, \end{aligned}$$

which forces $\mathcal{L}(E, T) = 1 + 3T + 9T^2$.

Functional equation

Corollary (of the Weil conjectures and root number results [3])

There is a global root number $\epsilon_E \in \{\pm 1\}$ *such that*

$$\mathcal{L}(E,T) = \epsilon_E q^{d_E} T^{d_E} \mathcal{L}(E, 1/q^2 T).$$

Furthermore, there is a simple algorithm to compute ϵ_E *in terms of the reduction type of* E *at each place in the support of* $\mathfrak{f}_E$*.*

If $\{b_i\}_{i=0}^{d_E}$ are the coefficients of $\mathcal{L}(E,T)$, then

$$\sum_{i=0}^{d_E} b_i T^i = \sum_{i=0}^{d_E} \epsilon_E b_i q^{d_E - 2i} T^{d_E - i} = \sum_{i=0}^{d_E} \epsilon_E b_{d_E - i} q^{2i - d_E} T^i,$$

so that b_i can be computed as $\epsilon_E b_{d_E - i} q^{2i - d_E}$ when $\lceil d_E/2 \rceil \le i \le d_E$.

[3] by the works of Deligne, Rohrlich, Kobayashi, and Imai

Quintic example

Let E be the elliptic curve $y^2 = x^3 + x^2 + t^4 + t^2$ over $K = \mathbb{F}_3(t)$. Then

$$\deg \mathcal{L}(E, T) = d_E = 4(0) - 4 + \deg(6[\tfrac{1}{t}] + [t] + [t^2 + 1]) = 5.$$

By computing $\mathcal{L}_v(E, T^{\deg v})^{-1}$ for all places v of K with $\deg v \leq 2$,

$$\mathcal{L}(E, T) \equiv 1 + 3T + 9T^2 \mod T^3,$$

which forces $\mathcal{L}(E, T) = 1 + 3T + 9T^2 + 27\epsilon_E T^3 + 81\epsilon_E T^4 + 243\epsilon_E T^5$.

In fact, $\epsilon_E = -1$, since $\epsilon_{E,t} = \epsilon_{E,t^2+1} = -1$ and

$$\begin{aligned}\epsilon_{E,\frac{1}{t}} &= -(\Delta_{E'}, a_{6,E'}) \cdot \left(\frac{v_{\frac{1}{t}}(a_{6,E'})}{3}\right)^{v_{\frac{1}{t}}(\Delta_{E'})} \cdot \left(\frac{-1}{3}\right)^{\frac{v_{\frac{1}{t}}(\Delta_{E'})(v_{\frac{1}{t}}(\Delta_{E'})-1)}{2}} \\ &= -1,\end{aligned}$$

where E' is the elliptic curve $y^2 = x^3 + (\frac{1}{t})^2 x^2 + (\frac{1}{t})^4 + (\frac{1}{t})^2$ over $K_{\frac{1}{t}}$.

ℓ-adic representations

In general, the formal L-function of an almost everywhere unramified ℓ-adic representation $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ is given by

$$\mathcal{L}(\rho, T) := \prod_v \frac{1}{\mathcal{L}_v(\rho, T^{\deg v})} \in \overline{\mathbb{Q}_\ell}[[T]],$$

where $\mathcal{L}_v(\rho, T)$ is defined similarly as before.

Corollary (of the Weil conjectures [4])

If ρ has no $G_{\overline{k}K}$-invariants, then $\mathcal{L}(\rho, T) \in \overline{\mathbb{Q}_\ell}[T]$ has degree

$$d_\rho := (2g_C - 2)\dim\rho + \deg \mathfrak{f}_\rho,$$

and satisfies the functional equation

$$\mathcal{L}(\rho, T) = \epsilon_\rho q^{d_\rho(\frac{w_\rho+1}{2})} T^{d_\rho} \mathcal{L}(\rho, 1/q^{w_\rho+1}T)^{\sigma_\rho},$$

where w_ρ is the weight of ρ and σ_ρ is some automorphism on $\overline{\mathbb{Q}_\ell}$.

[4] by the works of Grothendieck and Deligne

Magma implementation

I have implemented `intrinsics` for computing formal L-functions of arbitrary ℓ-adic representations with or without functional equations.

This includes specific examples of motives over $k(t)$:

- elliptic curves, with functional equation except when $\mathrm{char}(k) = 2, 3$
  - functional equation when $\mathrm{char}(k) = 2, 3$ require Hilbert symbols
  - faster than `LFunction()` when $\mathrm{char}(k) = 2, 3, 5, 7$
- Dirichlet characters, without functional equation
  - functional equation requires efficient computations of Gauss sums
  - non-square-free modulus is surprisingly tricky
- tensor products assuming their conductors are disjoint
  - degree computation requires $\mathfrak{f}_{\rho\otimes\tau}$ in terms of $\mathfrak{f}_\rho$ and $\mathfrak{f}_\tau$
  - functional equation requires $\epsilon_{\rho\otimes\tau}$ in terms of ϵ_ρ and ϵ_τ
- any other nice motives?
  - hyperelliptic curves?
  - Artin representations?