Wednesday, 18 January 2023

Introduction to abelian varieties over finite fields

Dual abelian varieties [1]

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[1] J S Milne (2008) Abelian Varieties

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Here $\mathrm{Cl}(E)$ is the **class group** of Weil divisors $\sum_{P \in E} n_P(P)$ modulo $\sim$, where $D \sim 0$ if D is the divisor (f) of some rational function $f \in \overline{K}(E)^\times$, and $\mathrm{Cl}^0(E)$ is its subgroup with $\sum_{P \in E} n_P = 0$.

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Idea: for any $D \in \mathrm{Cl}^0(E)$, the Riemann-Roch space $\mathrm{L}(D + (O))$, where

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For an elliptic curve E, its *dual* is $\mathrm{Cl}^0(E)$.

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If $f : Y \to X$ is a morphism, then there is also a **pull-back**

$$f^*\mathcal{L} := f^{-1}\mathcal{L} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \in \mathrm{Pic}(Y).$$

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This is a homomorphism, by **theorem of the square**

$$\tau_{a+b}^* \mathcal{L} \cdot \mathcal{L} \cong \tau_a^* \mathcal{L} \cdot \tau_b^* \mathcal{L}, \qquad a, b \in A(K).$$

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This follows from **theorem of the cube** [2] that

$$(f+g+h)^* \mathcal{L} \cdot (f+g)^* \mathcal{L}^{-1} \cdot (f+h)^* \mathcal{L}^{-1} \cdot (g+h)^* \mathcal{L}^{-1} \cdot f^* \mathcal{L} \cdot g^* \mathcal{L} \cdot h^* \mathcal{L}$$

is trivial for any regular maps $f, g, h : V \to A$ from a variety V/K.

[2] Theorem I.5.1

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In fact, if $\mathcal{L} \in \mathrm{Pic}(A)$ is ample, then $\ker(\lambda_{\mathcal{L}}) \leq A(K)$ is finite. [3]

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In fact, if $D \in \mathrm{Cl}(E)$ is effective, then $\deg D = 0$ iff $\lambda_{\mathcal{L}(D)} = 0$. [4]

[4] Example I.8.3

Translation-invariant invertible sheaves

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Need an abelian variety $\widehat{A}$ such that $\widehat{A}(K) \cong \mathrm{Pic}^0(A)$.

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Construction of dual abelian varieties

Idea: $\lambda_{\mathcal{L}} : A(K) \to \mathrm{Pic}^0(A)$ has kernel $\mathrm{K}(\mathcal{L})(K)$, and in fact is surjective if $\mathcal{L} \in \mathrm{Pic}(A)$ is ample, [6]

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Remark

Since $\mathcal{L} \in \mathrm{Pic}^0(A)$ iff $+^*\mathcal{L} \cong \pi_1^*\mathcal{L} \cdot \pi_2^*\mathcal{L}$, addition on A lifts to multiplication on $\mathcal{L}$ and makes $\mathcal{G}(\mathcal{L}) := \mathcal{L} \setminus \{0\}$ an abelian group scheme over K.

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[7] Proposition I.9.3

Representability of dual abelian varieties

Consider the functor $\mathcal{F} : \mathbf{Var}_K \to \mathbf{Set}$ that associates a variety V/K to the set of isomorphism classes of $\mathcal{L} \in \mathrm{Pic}(A \times V)$ such that

- $\mathcal{L}|_{A \times \{x\}} \in \mathrm{Pic}^0(A_x)$ for any $x \in V$, and
- $\mathcal{L}|_{\{0\} \times V} \cong \mathcal{O}_V$.

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- $\mathcal{L}|_{A \times \{x\}} \in \mathrm{Pic}^0(A_x)$ for any $x \in V$, and
- $\mathcal{L}|_{\{0\} \times V} \cong \mathcal{O}_V$.

Theorem

$\widehat{A}$ represents $\mathcal{F}$. In other words $\mathcal{F}(V) = \mathrm{Hom}(V, \widehat{A})$ for any variety V/K.

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Sketched in Section I.8. □

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This defines a *Weil pairing*

$$\mathrm{e}_\phi : \ker(\phi) \times \ker(\widehat{\phi}) \to \mu_n.$$

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A **polarisation** on A is an isogeny $\lambda : A \to \widehat{A}$ such that $\lambda = \lambda_{\mathcal{L}}$ over $\overline{K}$ for some ample $\mathcal{L} \in \mathrm{Pic}(A_{\overline{K}})$.

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$$(\phi + \psi)^\dagger = \phi^\dagger + \psi^\dagger, \qquad (\phi \circ \psi)^\dagger = \psi^\dagger \circ \phi^\dagger, \qquad \phi, \psi \in \mathrm{End}^0(A),$$

and $a^\dagger = a$ for any $a \in \mathbb{Q}$.

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