Denominators of BSD quotients

David Ang

London School of Geometry and Number Theory

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Mordell's theorem

Let E be an elliptic curve over $\mathbb{Q}$ given by a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \qquad a_i \in \mathbb{Q}.$$

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Its rational points forms a group $E(\mathbb{Q})$ under a geometric addition law.

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The **torsion subgroup** $\mathrm{tor}(E)$ is well understood.

Theorem (Mazur)

$$\mathrm{tor}(E) \cong \begin{cases} C_n & n = 1,2,3,4,5,6,7,8,9,10,12 \\ C_2 \oplus C_{2n} & n = 1,2,3,4 \end{cases}.$$

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The **rank** $\mathrm{rk}(E)$ is somewhat mysterious.

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Assume E has conductor N. The L-function of E is the infinite product

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where $a_p(E) := 1 + p - \#E(\mathbb{F}_p)$ and $\epsilon \in \{-1,0,1\}$.

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$$\mathrm{ord}_{s=1} L(E,s) = \mathrm{rk}(E).$$

This is known for $\mathrm{ord}_{s=1} L(E,s) \leq 1$. Assume that $\mathrm{ord}_{s=1} L(E,s) = 0$.

The Birch–Swinnerton-Dyer quotient

Conjecture (strong Birch–Swinnerton-Dyer)

$$\frac{L(E,1)}{\Omega(E)} = \frac{\mathrm{Tam}(E) \cdot \#\text{Ш}(E)}{\#\mathrm{tor}(E)^2}.$$

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The LHS is the **algebraic L-value** and the RHS is the **BSD quotient**.

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- The **Tate–Shafarevich group** is the finite group

$$\text{Ш}(E) := \ker\left(H^2(\mathbb{Q},E) \to H^2(\mathbb{R},E) \times \prod_p H^2(\mathbb{Q}_p,E)\right).$$

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where ω_E is the **Néron differential**.

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It is the least positive element of the *real period lattice* of E.

Elliptic curve with Cremona label 90c3 (LMFDB label 90.c7)

Introduction
Overview · Random · Universe · Knowledge

L-functions
Rational · All

Modular forms
Classical · Maass · Hilbert · Bianchi

Varieties
Elliptic curves over $\mathbb{Q}$
Elliptic curves over $\mathbb{Q}(\alpha)$
Genus 2 curves over $\mathbb{Q}$
Higher genus families
Abelian varieties over $\mathbb{F}_q$

Fields
Number fields
p-adic fields

Representations
Dirichlet characters
Artin representations

Groups
Galois groups
Sato-Tate groups

Database

Minimal Weierstrass equation

Show commands: Magma / Oscar / PariGP / SageMath

$$y^2 + xy + y = x^3 - x^2 - 122x + 1721$$

(homogenize, simplify)

Mordell-Weil group structure

$\mathbb{Z}/12\mathbb{Z}$

Torsion generators

$(-9, 49)$

Integral points

$(-15, 7)$, $(-9, 49)$, $(-9, -41)$, $(1, 39)$, $(1, -41)$, $(9, 31)$, $(9, -41)$, $(21, 79)$, $(21, -101)$, $(81, 679)$, $(81, -761)$

Invariants

Conductor:	90	=	$2 \cdot 3^2 \cdot 5$
Discriminant:	-1119744000	=	$-1 \cdot 2^{12} \cdot 3^7 \cdot 5^3$
j-invariant:	$-\frac{273359449}{1536000}$	=	$-1 \cdot 2^{-12} \cdot 3^{-1} \cdot 5^{-3} \cdot 11^3 \cdot 59^3$
Endomorphism ring:	$\mathbb{Z}$		
Geometric endomorphism ring:	$\mathbb{Z}$		(no potential complex multiplication)
Sato-Tate group:	$\mathrm{SU}(2)$		
Faltings height:	$0.42032494899046121656963281857\ldots$		
Stable Faltings height:	$-0.12898119534359362912798979989\ldots$		
abc quality:	$1.0491971880149842\ldots$		
Szpiro ratio:	$6.308958268204\ldots$		

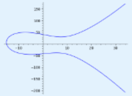
BSD invariants

Analytic rank:	0
Regulator:	1
Real period:	$1.3375959945886485057653424429\ldots$
Tamagawa product:	$144 = (2^2 \cdot 3) \cdot 2^2 \cdot 3$
Torsion order:	12
Analytic order of Ш:	1 (exact)
Special value:	$L(E,1) \approx 1.3375959945886485057653424429$

$$\frac{L(E,1)}{\Omega(E)} = \frac{\mathrm{Tam}(E) \cdot \#Ш(E)}{\#\mathrm{tor}(E)^2}$$

Properties

Label	90c3



Conductor	90
Discriminant	-1119744000
j-invariant	$-\frac{273359449}{1536000}$
CM	no
Rank	0
Torsion structure	$\mathbb{Z}/12\mathbb{Z}$

Related objects
Isogeny class 90c
Minimal quadratic twist 30a3
All twists
L-function
Symmetric square L-function
Modular form 90.2.a.c

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Denominator bounds

Observe that BSD quotients have bounded denominators.

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Theorem (Mazur)

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Corollary

$$\mathrm{ord}_p\left(\frac{\mathrm{Tam}(E)\cdot \#\text{Ш}(E)}{\#\mathrm{tor}(E)^2}\right) \geq \begin{cases} -8 & \textit{if } p = 2 \\ -4 & \textit{if } p = 3 \\ -2 & \textit{if } p = 5,7 \\ 0 & \textit{if } p \geq 11 \end{cases}.$$

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There are typically cancellations between $\mathrm{tor}(E)$ and $\mathrm{Tam}(E)$.

Torsion cancellations

Theorem (Lorenzini, 2010)

Assume that $\mathrm{tor}(E)$ *has a point of order* $n \geq 4$*.*

- *If* $n = 4$*, then* $2 \mid \mathrm{Tam}(E)$*, except for 15a7, 15a8, 17a4.*
- *If* $n \geq 5$*, then* $n \mid \mathrm{Tam}(E)$*, except for 11a3, 14a4, 14a6, 20a2.*
- *If* $n = 9$*, then* $27 \mid \mathrm{Tam}(E)$*.*

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Corollary

With seven exceptions,

$$\mathrm{ord}_p\left(\frac{\mathrm{Tam}(E)\cdot\#Ш(E)}{\#\mathrm{tor}(E)^2}\right) \geq \begin{cases} -5 & \text{if } p = 2 \text{ and } \mathrm{tor}(E) \cong C_2 \oplus C_{2n} \\ -3 & \text{if } p = 2 \text{ and } \mathrm{tor}(E) \ncong C_2 \oplus C_{2n} \\ -2 & \text{if } p = 3 \text{ and } \mathrm{tor}(E) \cong C_3 \\ -1 & \text{if } p = 3 \text{ and } \mathrm{tor}(E) \ncong C_3 \\ -1 & \text{if } p = 5, 7 \\ 0 & \text{if } p \geq 11 \end{cases}.$$

The seven exceptions

Let $\mathrm{BSD}(E)$ denote the BSD quotient.

E	$11a3$	$14a4$	$14a6$	$15a7$	$15a8$	$17a4$	$20a2$
$\mathrm{tor}(E)$	C_5	C_6	C_6	C_4	C_4	C_4	C_6
$\mathrm{Tam}(E)$	1	2	2	1	1	1	3
$Ш(E)$	1	1	1	1	1	1	1
$\mathrm{BSD}(E)$	$\frac{1}{5^2}$	$\frac{1}{2\cdot 3^2}$	$\frac{1}{2\cdot 3^2}$	$\frac{1}{2^4}$	$\frac{1}{2^4}$	$\frac{1}{2^4}$	$\frac{1}{2^2\cdot 3}$

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$Ш(E)$	1	1	1	1	1	1	1
$\mathrm{BSD}(E)$	$\frac{1}{5^2}$	$\frac{1}{2\cdot 3^2}$	$\frac{1}{2\cdot 3^2}$	$\frac{1}{2^4}$	$\frac{1}{2^4}$	$\frac{1}{2^4}$	$\frac{1}{2^2\cdot 3}$
$c_0(E)$	5	3	3	2	4	4	2
$c_0(E)\mathrm{BSD}(E)$	$\frac{1}{5}$	$\frac{1}{2\cdot 3}$	$\frac{1}{2\cdot 3}$	$\frac{1}{2^3}$	$\frac{1}{2^2}$	$\frac{1}{2^2}$	$\frac{1}{2\cdot 3}$

Here, $c_0(E)$ is the **Manin constant** in the LMFDB.

Elliptic curve with Cremona label 11a3 (LMFDB label 11.a3)

BSD invariants

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Analytic rank:	0
Regulator:	1
Real period:	$6.3460465213977671084439730838\ldots$
Tamagawa product:	1
Torsion order:	5
Analytic order of Ш:	1 (exact)
Special value:	$L(E,1) \approx 0.25384186085591068433775892335$

BSD formula

$$0.253841861 \approx L(E,1) = \frac{\#Ш(E/\mathbb{Q}) \cdot \Omega_E \cdot \mathrm{Reg}(E/\mathbb{Q}) \cdot \prod_p c_p}{\#E(\mathbb{Q})_{\mathrm{tor}}^2} \approx \frac{1 \cdot 6.346047 \cdot 1.000000 \cdot 1}{5^2} \approx 0.253841861$$

Modular invariants

Modular form 11.2.a.a

$$q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} + q^{11} - 2q^{12} + 4q^{13} + 4q^{14} - q^{15} - 4q^{16} - 2q^{17} + 4q^{18} + O(q^{20})$$

For more coefficients, see the Downloads section to the right.

Modular degree:	5
$\Gamma_0(N)$-optimal:	no
Manin constant:	5

Local data

This elliptic curve is semistable. There is only one prime of bad reduction:

prime	Tamagawa number	Kodaira symbol	Reduction type	Root number	ord(N)	ord(Δ)	ord(j)_
11	1	I_1	Split multiplicative	-1	1	1	1

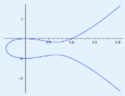
Galois representations

The ℓ-adic Galois representation has maximal image for all primes ℓ except those listed in the table below.

prime ℓ	mod-ℓ image	ℓ-adic image
5	5B.1.1	25.120.0.1

Properties

Label	11a3



Conductor	11
Discriminant	-11
j-invariant	$-\frac{4096}{11}$
CM	no
Rank	0
Torsion structure	$\mathbb{Z}/5\mathbb{Z}$

Related objects

Isogeny class 11a
Minimal quadratic twist 11a3
All twists
L-function
Symmetric square L-function
Symmetric cube L-function
Modular form 11.2.a.a

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The Manin constant

Theorem (Modularity, version L)

There is an eigenform $f_E \in S_2(\Gamma_0(N))$ with eigenvalues $a_p(E)$ such that

$$L(f_E, s) = L(E, s).$$

In particular, this defines a differential $f_E(q)\mathrm{d}q$ on $X_0(N)$.

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There is a finite morphism $\phi_E : X_0(N) \twoheadrightarrow E$ defined over $\mathbb{Q}$ such that

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for some positive integer $c_0(E)$.

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Conjecturally $c_0(E) = 1$ for all $\Gamma_0(N)$-*optimal* elliptic curves (known in the semistable case!), but the seven exceptions are not $\Gamma_0(N)$-optimal.

A refined conjecture

Conjecture

With no exceptions,

$$\operatorname{ord}_p\left(\frac{c_0(E)\cdot \mathrm{Tam}(E)\cdot \#Ш(E)}{\#\mathrm{tor}(E)^2}\right) \geq \begin{cases} -3 & p = 2 \\ -1 & p = 3, 5, 7 \\ 0 & p \geq 11 \end{cases}.$$

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Conjecture

Assume that $\mathrm{tor}(E) \cong C_3$*. Then* $3 \mid c_0(E)\cdot \mathrm{Tam}(E)\cdot \#Ш(E)$*.*

I can prove this under the strong Birch–Swinnerton-Dyer conjecture.

Modular symbols

If $f \in S_2(\Gamma_0(N))$ and $p \nmid N$, the Hecke operator T_p acts on periods by

$$(1 + p - T_p) \cdot \int_0^{\infty} f(q)\mathrm{d}q = \sum_{a=1}^{p-1} \int_0^{\frac{a}{p}} f(q)\mathrm{d}q.$$

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If $f = f_E$ and p is odd, this says that

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If the strong Birch–Swinnerton-Dyer conjecture holds,

$$(1 + p - a_p(E)) \cdot \frac{c_0(E) \cdot \mathrm{Tam}(E) \cdot \#\text{Ш}(E)}{\#\mathrm{tor}(E)^2} \in \mathbb{Z}.$$

Modular symbols

If $f \in S_2(\Gamma_0(N))$ and $p \nmid N$, the Hecke operator T_p acts on periods by

$$(1 + p - T_p) \cdot \int_0^\infty f(q) \mathrm{d}q = \sum_{a=1}^{p-1} \int_0^{\frac{a}{p}} f(q) \mathrm{d}q.$$

If $f = f_E$ and p is odd, this says that

$$(1 + p - a_p(E)) \cdot (-L(E, 1)) = \frac{\Omega(E)}{c_0(E)} \cdot n, \quad n \in \mathbb{Z}.$$

If the strong Birch–Swinnerton-Dyer conjecture holds,

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If $\mathrm{tor}(E) \cong C_3$, it suffices to find an odd prime $p \nmid N$ such that

$$1 + p - a_p(E) \equiv 3 \mod 9.$$

3-adic Galois images

In terms of $\rho_{E,3} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_3)$,

$$p = \det(\rho_{E,3}(\mathrm{Fr}_p)), \qquad a_p(E) = \mathrm{tr}(\rho_{E,3}(\mathrm{Fr}_p)).$$

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Theorem (Rouse–Sutherland–Zureick-Brown, 2022)

Assume that $\mathrm{tor}(E) \cong C_3$. Then $\mathrm{im}(\rho_{E,3})$ is one of the explicit matrix subgroups 3.8.0.1, 3.24.0.1, 9.24.0.1/2, 9.72.0.1/2/3/4/6/7/8/9/10, 27.72.0.1, 27.648.13.25, 27.648.18.1, or 27.1944.55.31/37/43/44.

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Each $\mathrm{im}(\rho_{E,3})$ contains a matrix M such that $3 = 1 + \det(M) - \mathrm{tr}(M)$, except for 9.72.0.1, but Tate's algorithm shows $3 \mid \mathrm{Tam}(E)$ in this case.

Concluding remarks

Theorem (A., 2023)

Assume the 3-part of the strong Birch–Swinnerton-Dyer conjecture. Then

$$\mathrm{ord}_p\left(\frac{c_0(E)\cdot \mathrm{Tam}(E)\cdot \#\text{Ш}(E)}{\#\mathrm{tor}(E)^2}\right) \geq \begin{cases} -3 & p=2 \\ -1 & p=3,5,7 \\ 0 & p\geq 11 \end{cases}.$$

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Does this generalise to $\mathbb{F}_q(C)$ or $\mathrm{ord}_{s=1}L(E,s)\geq 1$?