Division polynomials of elliptic curves

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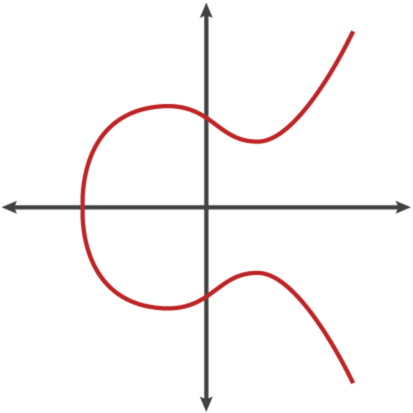
Elliptic curves

An elliptic curve over a field F is a smooth projective curve E of genus one, equipped with a fixed point $\mathcal{O}$ defined over F.



They are one of the simplest non-trivial objects in arithmetic geometry.

Weierstrass equations

In `mathlib`, an **elliptic curve** E over an integral domain R is a tuple $(a_1, a_2, a_3, a_4, a_6) \in R^5$, with an extra condition that $\Delta \in R^\times$, where

$$
\begin{aligned}
b_2 &:= a_1^2 + 4a_2, \\
b_4 &:= 2a_4 + a_1 a_3, \\
b_6 &:= a_3^2 + 4a_6, \\
b_8 &:= a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2, \\
\Delta &:= -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6.
\end{aligned}
$$

A **point** on E is either $\mathcal{O}$ or an **affine point** $(x, y)_a \in R^2$ such that

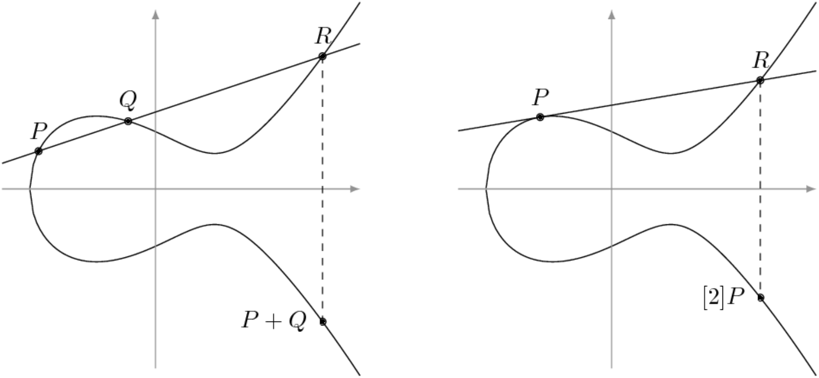
$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x^3 + a_6,$$

so the points on E vanish on the polynomial $\mathcal{E} \in R[X, Y]$ given by

$$\mathcal{E} := Y^2 + a_1XY + a_3Y - (X^3 + a_2X^2 + a_4X + a_6).$$

Group law

The points on E can be endowed with a geometric addition law.



In 2023, we formalised a novel algebraic proof of the group law on E.

Is there an explicit formula for $[n]P$ in terms of P?

An impossible exercise

The Arithmetic of Elliptic Curves by Silverman gives an answer.

Exercise (3.7(d))

Let $n \in \mathbb{Z}$. Prove that for any point $(x, y)_a$ on E,

$$[n](x, y)_a = \left(\frac{\phi_n(x, y)}{\psi_n(x, y)^2}, \frac{\omega_n(x, y)}{\psi_n(x, y)^3} \right)_a .$$

Silverman gives inductive definitions for $\phi_n, \omega_n, \psi_n \in F[X, Y]$.

This formula leads to a proof that

$$T_p E_{\overline{F}} \cong \begin{cases} \mathbb{Z}_p^2 & \text{if } \operatorname{char}(F) \neq p, \\ 0 \text{ or } \mathbb{Z}_p & \text{if } \operatorname{char}(F) = p. \end{cases}$$

These polynomials also feature in Schoof's algorithm.

Multiplication by 2

If $(x, y)_a$ is a generic affine point on E, then

$$[2](x, y)_a = \left(\frac{\phi_2(x, y)}{\psi_2(x, y)^2}, \frac{\omega_2(x, y)}{\psi_2(x, y)^3} \right)_a ,$$

where $\phi_2, \omega_2, \psi_2 \in F[X, Y]$ are given by

$$\begin{aligned} \psi_2 &:= 2Y + a_1 X + a_3, \\ \phi_2 &:= X\psi_2^2 - \bigcirc, \\ \omega_2 &:= \frac{1}{2}(\triangle - (a_1\phi_2 + a_3)\psi_2^3), \end{aligned}$$

for some $\bigcirc, \triangle \in F[X]$. If $(x, y)_a$ is a 2-torsion affine point on E, then

$$(x, y)_a = -(x, y)_a = (x, -y - a_1 x - a_3)_a,$$

so $\psi_2(x, y) = 2y + a_1 x + a_3 = 0$.

Projective coordinates

Let $(x, y)_a$ be an affine point on E. In **projective** coordinates,

$$[2](x, y)_a = (\phi_2(x, y)\psi_2(x, y) : \omega_2(x, y) : \psi_2(x, y)^3)_p.$$

In `mathlib`, a **projective point** on E is a class of $(x, y, z) \in F^3$ such that

$$y^2 z + a_1 xyz + a_3 yz^2 = x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3.$$

The point at infinity on E is $(0 : 1 : 0)_p$.

More naturally, in **Jacobian** coordinates with weights $(2 : 3 : 1)$,

$$[2](x, y)_a = (\phi_2(x, y) : \omega_2(x, y) : \psi_2(x, y))_j.$$

In `mathlib`, a **Jacobian point** on E is a class of $(x, y, z) \in F^3$ such that

$$y^2 + a_1 xyz + a_3 yz^3 = x^3 + a_2 x^2 z^2 + a_4 xz^4 + a_6 z^6.$$

The point at infinity on E is $(1 : 1 : 0)_j$.

Multiplication by n

Exercise (3.7(d), corrected)

Let $n \in \mathbb{Z}$. Prove that for any point $(x, y)_a$ on E,

$$[n](x, y)_a = (\phi_n(x, y) : \omega_n(x, y) : \psi_n(x, y))_j.$$

If $(x : y : z)_j$ is a point on E, then $x = y = 1$ whenever $z = 0$, so

$$\ker[n] = \{\mathcal{O}\} \cup \{(x, y)_a : \psi_n(x, y) = 0\}.$$

Conjecture

No one has done Exercise 3.7(d) purely inductively.

Xu gave a complete answer to this exercise and formalised it in Lean.

I will define ϕ_n, ω_n, ψ_n, and their auxiliary polynomials.

The polynomials ψ_n

The n-th **division polynomial** $\psi_n \in R[X, Y]$ is given by

$$\begin{aligned}
\psi_0 &:= 0, \\
\psi_1 &:= 1, \\
\psi_2 &:= 2Y + a_1 X + a_3, \\
\psi_3 &:= \bigcirc \\
&\qquad \text{where } \bigcirc := 3X^4 + b_2 X^3 + 3b_4 X^2 + 3b_6 X + b_8, \\
\psi_4 &:= \psi_2 \triangle \\
&\qquad \text{where } \triangle := 2X^6 + b_2 X^5 + 5b_4 X^4 + 10b_6 X^3 + 10b_8 X^2 + (b_2 b_8 - b_4 b_6) X + (b_4 b_8 - b_6^2), \\
\psi_{2n+1} &:= \psi_{n+2} \psi_n^3 - \psi_{n-1} \psi_{n+1}^3, \\
\psi_{2n} &:= \frac{\psi_{n-1}^2 \psi_n \psi_{n+2} - \psi_{n-2} \psi_n \psi_{n+1}^2}{\psi_2}, \\
\psi_{-n} &:= -\psi_n.
\end{aligned}$$

In `mathlib`, ψ_n is defined in terms of $\Psi_n \in R[X]$.

The polynomials Ψ_n

The polynomial $\Psi_n \in R[X]$ is given by

$$\begin{aligned}
\Psi_0 &:= 0, \\
\Psi_1 &:= 1, \\
\Psi_2 &:= 1, \\
\Psi_3 &:= \bigcirc, \\
\Psi_4 &:= \triangle, \\
\Psi_{2n+1} &:= \begin{cases} \Psi_{n+2}\Psi_n^3 - \square^2\Psi_{n-1}\Psi_{n+1}^3 & \text{if } n \text{ is odd,} \\ \square^2\Psi_{n+2}\Psi_n^3 - \Psi_{n-1}\Psi_{n+1}^3 & \text{if } n \text{ is even,} \end{cases} \\
&\quad \text{where } \square := 4X^3 + b_2X^2 + 2b_4X + b_6, \\
\Psi_{2n} &:= \Psi_{n-1}^2\Psi_n\Psi_{n+2} - \Psi_{n-2}\Psi_n\Psi_{n+1}^2, \\
\Psi_{-n} &:= -\Psi_n.
\end{aligned}$$

Then $\psi_n = \Psi_n$ when n is odd and $\psi_n = \psi_2\Psi_n$ when n is even.

The polynomials ϕ_n and Φ_n

In the coordinate ring of E,

$$\begin{aligned}\psi_2^2 &= (2Y + a_1X + a_3)^2 \\ &= 4(Y^2 + a_1XY + a_3Y) + a_1^2X^2 + 2a_1a_3X + a_3^2 \\ &\equiv \underbrace{4X^3 + b_2X^2 + 2b_4X + b_6}_{\square} \mod \mathcal{E}.\end{aligned}$$

In particular, ψ_n^2 and $\psi_{n+1}\psi_{n-1}$ are congruent to polynomials in $R[X]$.

The polynomial $\phi_n \in R[X, Y]$ is given by

$$\phi_n := X\psi_n^2 - \psi_{n+1}\psi_{n-1},$$

so that $\phi_n \equiv \Phi_n \mod \mathcal{E}$, where $\Phi_n \in R[X]$ is given by

$$\Phi_n := \begin{cases} X\Psi_n^2 - \square\Psi_{n+1}\Psi_{n-1} & \text{if } n \text{ is odd,} \\ X\square\Psi_n^2 - \Psi_{n+1}\Psi_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

The polynomials ω_n

The polynomial $\omega_n \in R[X, Y]$ is given by

$$\omega_n := \frac{1}{2}\left(\frac{\psi_{2n}}{\psi_n} - a_1\phi_n\psi_n - a_3\psi_n^3\right).$$

Lemma (Xu)

Let $n \in \mathbb{Z}$. Then $\psi_{2n}/\psi_n - a_1\phi_n\psi_n - a_3\psi_n^3$ is divisible by 2 in $\mathbb{Z}[a_i, X, Y]$.

Example ($a_1 = a_3 = 0$)

$$\omega_2 = \frac{\Psi_4}{2} = \frac{2X^6+4a_2X^5+10a_4X^4+40a_6X^3+10b_8X^2+(4a_2b_8-8a_4a_6)X+(2a_4b_8-16a_6^2)}{2}.$$

Define ω_n as the image of the quotient under $\mathbb{Z}[a_i, X, Y] \to R[X, Y]$.

When $n = 4$, this quotient has 15,049 terms.

Elliptic divisibility sequences

Integrality relies on the fact that ψ_n is an **elliptic divisibility sequence**.

Exercise (3.7(g))

For all $n, m, r \in \mathbb{Z}$, prove that $\psi_n \mid \psi_{nm}$ and

$$\psi_{n+m}\psi_{n-m}\psi_r^2 = \psi_{n+r}\psi_{n-r}\psi_m^2 - \psi_{m+r}\psi_{m-r}\psi_n^2.$$

Note that this generalises the recursive definitions of ψ_{2n+1} and ψ_{2n}.

Surprisingly, this needs the stronger result that ψ_n is an **elliptic net**.

Theorem (Xu)

Let $n, m, r, s \in \mathbb{Z}$. Then

$$\psi_{n+m}\psi_{n-m}\psi_{r+s}\psi_{r-s} = \psi_{n+r}\psi_{n-r}\psi_{m+s}\psi_{m-s} - \psi_{m+r}\psi_{m-r}\psi_{n+s}\psi_{n-s}.$$

Xu gave an elegant proof of this on Math Stack Exchange.

The Somos-4 invariant

As an elliptic sequence, ψ_n satisfies the **Somos-4 recurrence**

$$\psi_{n+2}\psi_{n-2} = \psi_2^2\psi_{n+1}\psi_{n-1} - \psi_3\psi_n^2.$$

An easy induction gives an invariant

$$\mathcal{I}(n) := \frac{\psi_{n-1}^2\psi_{n+2} + \psi_{n-2}\psi_{n+1}^2 + \psi_2^2\psi_n^3}{\psi_{n+1}\psi_n\psi_{n-1}}.$$

When $n = 2$, an explicit computation gives

$$\mathcal{I}(2) = \frac{\psi_4 + \psi_2^5}{\psi_3\psi_2} \equiv a_1\psi_2 \mod 2.$$

Being an invariant means that $\mathcal{I}(n) = \mathcal{I}(2)$, so

$$\frac{\psi_{n-1}^2\psi_{n+2} + \psi_{n-2}\psi_{n+1}^2 + \psi_2^2\psi_n^3}{\psi_2} \equiv a_1\psi_{n+1}\psi_n\psi_{n-1} \mod 2.$$

Integrality of ω_n

In particular,

$$\begin{aligned}\frac{\psi_{2n}}{\psi_n} &= \frac{\psi_{n-1}^2\psi_{n+2} - \psi_{n-2}\psi_{n+1}^2}{\psi_2} && \text{by definition of } \psi_{2n} \\ &\equiv \psi_2\psi_n^3 + a_1\psi_{n+1}\psi_n\psi_{n-1} \mod 2 && \text{by } \mathcal{I}(n) = \mathcal{I}(2) \\ &= 2Y\psi_n^3 + a_1\underbrace{(X\psi_n^2 + \psi_{n+1}\psi_{n-1})}_{\phi_n}\psi_n + a_3\psi_n^3 && \text{by definition of } \psi_2.\end{aligned}$$

Thus $\psi_{2n}/\psi_n - a_1\phi_n\psi_n - a_3\psi_n^3 \equiv 0 \mod 2$. In Lean,

$$\begin{aligned}\omega_n := \frac{\psi_{n+1}\psi_n\psi_{n-1}}{\psi_2\psi_3}(4\mathcal{E}(2\mathcal{E} + \square) + \psi_3(a_1\psi_2 - \tfrac{\partial\mathcal{E}}{\partial X})) \\ - \frac{\psi_{n-2}\psi_{n+1}^2}{\psi_2} + (Y - \psi_2)\psi_n^3,\end{aligned}$$

which is well-defined, since ψ_n is a divisibility sequence.

Other formalised results

The polynomial $\Psi_n^{(2)} \in R[X]$ is given by

$$\Psi_n^{(2)} := \begin{cases} \Psi_n^2 & \text{if } n \text{ is odd}, \\ \Box\Psi_n^2 & \text{if } n \text{ is even}, \end{cases}$$

so that $\Psi_2^{(2)} = \Box$ and $\Psi_n^{(2)} \equiv \psi_n^2 \mod \mathcal{E}$.

Exercise (3.7(b))

Show that $\Phi_n = X^{n^2} + \ldots$ and $\Psi_n^{(2)} = n^2 X^{n^2-1} + \ldots$.

This is an inductive computation of `natDegree` and `leadingCoeff`.

Exercise (3.7(c))

Prove that Φ_n and $\Psi_n^{(2)}$ are relatively prime.

Surprisingly, this needs Exercise 3.7(d) and the assumption that $\Delta \neq 0$.