London School of Geometry and Number Theory

London Learning Lean

Elliptic curves and the Mordell-Weil theorem

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Overview

- Introduction
- Abstract definition
- Concrete definition
- Implementation
- Associativity
- The Mordell-Weil theorem
- Selmer groups
- Future

Introduction — informally

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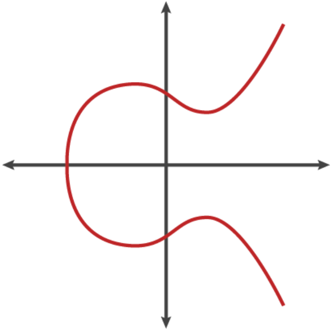

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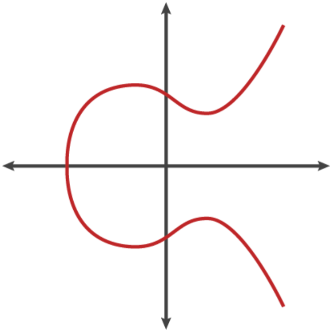

- A group — notion of addition of points!

Introduction — applications

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  - But modularity theorem — rational elliptic curves are modular!
- Distribution of ranks of rational elliptic curves.
  - The BSD conjecture — analytic rank equals algebraic rank?

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Good for algebraic geometry, but not very friendly...

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Group law is free, but still need equations...

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An elliptic curve E over a field F is a projective plane curve

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with $\Delta \neq 0$. [4]

[4] $\Delta := -(a_1^2+4a_2)^2(a_1^2a_6+4a_2a_6-a_1a_3a_4+a_2a_3^2-a_4^2)-8(2a_4+a_1a_3)^3-27(a_3^2+4a_6)^2+9(a_1^2+4a_2)(2a_4+a_1a_3)(a_3^2+4a_6)$

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Note the unique **point at infinity** when $Z = 0$! Call this point 0.

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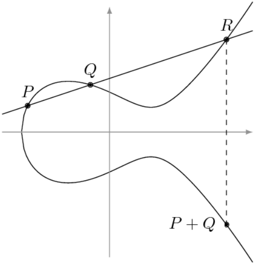

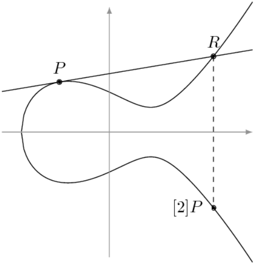

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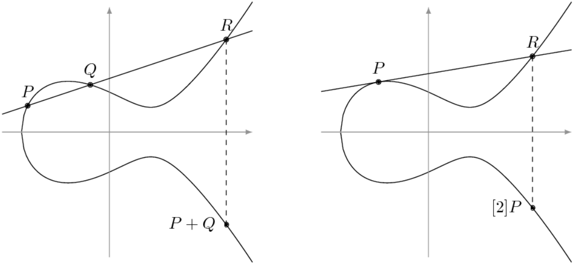

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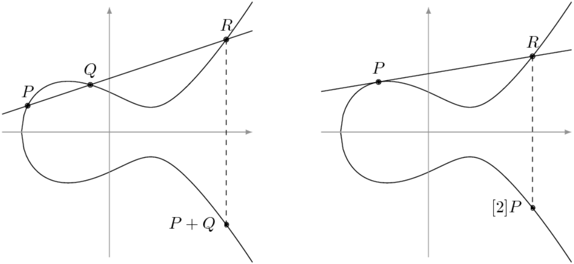

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Many cases... but all completely explicit!

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Three definitions of elliptic curves:

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Generality: 1. $\supset$ 2. $\stackrel{\text{RR}}{=}$ 3.

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def disc_aux {R : Type} [comm_ring R] (a₁ a₂ a₃ a₄ a₆ : R) : R :=
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structure EllipticCurve (R : Type) [comm_ring R] :=
 (a₁ a₂ a₃ a₄ a₆ : R) (disc : units R) (disc_eq : disc.val = disc_aux a₁ a₂ a₃ a₄ a₆)
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```

This is the *curve* E — what about the *group* $E(K)$?

Implementation — the group

```
variables {F : Type} [field F] (E : EllipticCurve F) (K : Type) [field K] [algebra F K]

inductive point
  | zero
  | some (x y : K) (w : y^2 + E.a₁*x*y + E.a₃*y = x^3 + E.a₂*x^2 + E.a₄*x + E.a₆)

notation E(K) := point E K
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instance : has_zero E(K) := ⟨zero⟩
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- Negation is easy.

```
def neg : E(K) → E(K)
  | zero := zero
  | (some x y w) := some x (−y − E.a₁*x − E.a₃) $
    begin
      rw [← w],
      ring
    end

instance : has_neg E(K) := ⟨neg⟩
```

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- Addition is complicated...

```
def add : E(K) → E(K) → E(K)
  | zero P := P
  | P zero := P
  | (some x1 y1 w1) (some x2 y2 w2) :=
    if x_ne : x1 ≠ x2 then                          -- add distinct points
     let L := (y1 - y2) / (x1 - x2),
         x3 := L^2 + E.a1*L - E.a2 - x1 - x2,
         y3 := -L*x3 - E.a1*x3 - y1 + L*x1 - E.a3
     in some x3 y3 $ by { ... }
    else if y_ne : y1 + y2 + E.a1*x2 + E.a3 ≠ 0 then -- double a point
     ...
    else                                            -- draw vertical line
     zero

instance : has_add E(K) := ⟨add⟩
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- Commutativity is... doable.

```
lemma add_comm (P Q : E(K)) : P + Q = Q + P :=
 begin
   rcases ⟨P, Q⟩ with ⟨_ | _, _ | _⟩,
   ... -- six cases
 end
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- Associativity is... impossible?

```
lemma add_assoc (P Q R : E(K)) : (P + Q) + R = P + (Q + R) :=
 begin
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  ... -- ??? cases
 end
```

Associativity — explaining the problem

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    - Probably(?) times out with 130,000(!) coefficients.

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- Proof in Coq (by Evmorfia-Iro Bartzia and Pierre-Yves Strub [6]) that $E(K) \cong \mathrm{Pic}^0_{E/F}(K)$ but only for $\mathrm{char}\ F \neq 2, 3$.

[6] A Formal Library for Elliptic Curves in the Coq Proof Assistant (2015)

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- Functoriality $\mathbf{Alg}_F \to \mathbf{Ab}$.

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def point_hom (φ : K →ₐ[F] L) : E(K) → E(L)
  | zero := zero
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lemma point_hom.id (P : E(K)) : point_hom (K→[F]K) P = P

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```

- Galois module structure $\mathrm{Gal}(L/K) \curvearrowright E(L)$.

```
def point_gal (σ : L ≃ₐ[K] L) : E(L) → E(L)
  | zero := zero
  | (some x y w) := some (σ · x) (σ · y) $ by { . . . }

variables [finite_dimensional K L] [is_galois K L]

lemma point_gal.fixed :
  mul_action.fixed_points (L ≃ₐ[K] L) E(L) = (point_hom (K→[F]L)).range
```

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Modulo associativity, what has been done?

- Isomorphisms $(x, y) \mapsto (u^2 x + r, u^3 y + u^2 s x + t)$.

```
variables (u : units F) (r s t : F)

def cov : EllipticCurve F :=
{ a₁ := u.inv*(E.a₁ + 2*s),
  a₂ := u.inv^2*(E.a₂ − s*E.a₁ + 3*r − s^2),
  a₃ := u.inv^3*(E.a₃ + r*E.a₁ + 2*t),
  a₄ := u.inv^4*(E.a₄ − s*E.a₃ + 2*r*E.a₂ − (t + r*s)*E.a₁ + 3*r^2 − 2*s*t),
  a₆ := u.inv^6*(E.a₆ + r*E.a₄ + r^2*E.a₂ + r^3 − t*E.a₃ − t^2 − r*t*E.a₁),
  disc := ⟨u.inv^12*E.disc.val, u.val^12*E.disc.inv, by { ... }, by { ... }⟩,
  disc_eq := by { simp only, rw [disc_eq, disc_aux, disc_aux], ring } }

def cov.to_fun : (E.cov u r s t)(K) → E(K)
  | zero := zero
  | (some x y w) := some (u.val^2*x + r) (u.val^3*y + u.val^2*s*x + t) $ by { ... }

def cov.inv_fun : E(K) → (E.cov u r s t)(K)
  | zero := zero
  | (some x y w) := some (u.inv^2*(x − r)) (u.inv^3*(y − s*x + r*s − t)) $ by { ... }

def cov.equiv_add : (E.cov u r s t)(K) ≃+ E(K) :=
  ⟨cov.to_fun u r s t, cov.inv_fun u r s t, by { ... }, by { ... }, by { ... }⟩
```

Associativity — ignoring the problem

Modulo associativity, what has been done?

- 2-division polynomial $\psi_2(x)$.

```
def ψ₂_x : cubic K := ⟨4, E.a₁^2 + 4*E.a₂, 4*E.a₄ + 2*E.a₁*E.a₃, E.a₃^2 + 4*E.a₆⟩

lemma ψ₂_x.disc_eq_disc : (ψ₂_x E K).disc = 16*E.disc
```

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- Structure of $E(K)[2]$.

```
notation E(K)[n] := ((·) n : E(K) →+ E(K)).ker

lemma E₂.x {x y w} : some x y w ∈ E(K)[2] ↔ x ∈ (ψ₂_x E K).roots

theorem E₂.card_le_four : fintype.card E(K)[2] ≤ 4

variables [algebra ((ψ₂_x E F).splitting_field) K]

theorem E₂.card_eq_four : fintype.card E(K)[2] = 4

lemma E₂.gal_fixed (σ : L ≃ₐ[K] L) (P : E(L)[2]) : σ • P = P
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The Mordell-Weil theorem — statement and proof

Theorem (Mordell-Weil)

Let K be a number field. Then $E(K)$ is finitely generated.

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Three steps.

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The descent step is done (Jujian Zhang).

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def covm.equiv_add : (E.cov _ _ _ _)(K) ≃+ E(K) := cov.equiv_add 1 0 (−E.a₁/2) (−E.a₃/2)
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```
variables [finite_dimensional K L] [is_galois K L] (n : ℕ)

lemma range_le_comap_range : n·E(K) ≤ add_subgroup.comap (point_hom _) n·E(L)

def Φ : add_subgroup E(K)/n :=
 (quotient_add_group.map _ _ _ $ range_le_comap_range n).ker

lemma Φ_mem_range (P : Φ n E L) : point_hom _ P.val.out' ∈ n·E(L)

def κ : Φ n E L → L ≃ₐ[K] L → E(L)[n] :=
 λ P σ, ⟨σ · (Φ_mem_range n P).some − (Φ_mem_range n P).some, by { . . . }⟩

lemma κ.injective : function.injective $ κ n

def coker_2_of_fg_extension.fintype : fintype E(L)/2 → fintype E(K)/2
```

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```
variables (ha₁ : E.a₁ = 0) (ha₃ : E.a₃ = 0) (h3 : (ψ₂_x E K).roots = {e₁, e₂, e₃})

def δ : E(K) → (units K) / (units K)^2 × (units K) / (units K)^2
  | zero := 1
  | (some x y w) :=
   if he₁ : x = e₁ then
    (units.mk0 ((e₁ - e₃) / (e₁ - e₂)) $ by { . . . }, units.mk0 (e₁ - e₂) $ by { . . . })
   else if he₂ : x = e₂ then
    (units.mk0 (e₂ - e₁) $ by { . . . }, units.mk0 ((e₂ - e₃) / (e₂ - e₁)) $ by { . . . })
   else
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- Prove $\ker \delta = 2E(K)$.

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- Reduce to $E[2] \subset E(K)$.
- Define a **complete** 2-**descent** homomorphism

$$\delta \ : \ E(K) \quad \longrightarrow \quad K^\times/(K^\times)^2 \times K^\times/(K^\times)^2.$$

- Prove $\ker \delta = 2E(K)$.

Here $\supseteq$ is obvious, while $\subseteq$ is long but constructive.

```
lemma δ.ker : (δ ha₁ ha₃ h3).ker = 2·E(K) :=
 begin
  . . . -- completely constructive proof
 end
```

The Mordell-Weil theorem — weak Mordell-Weil

Prove that $E(K)/2E(K)$ is finite with **complete** 2-**descent**.

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  Here S is a finite set of "ramified" places of K.

```
lemma δ.range_le : (δ ha₁ ha₃ h3).range ≤ K(S, 2) × K(S, 2) := sorry -- ramification theory?
```

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  with kernel $K(\emptyset, n)$. Thus

$$K(S,n) \text{ finite} \iff K(\emptyset,n) \text{ finite.}$$

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```
def selmer : subgroup $ (units K) / (units K)^n :=
 { carrier := {x | ∀ p ∉ S, val_of_ne_zero_mod p x = 1},
   one_mem' := by { ... },
   mul_mem' := by { ... },
   inv_mem' := by { ... } }

notation K(S, n) := selmer K S n

def selmer.val : K(S, n) →* S → multiplicative (zmod n) :=
 { to_fun := λ x p, val_of_ne_zero_mod p x,
   map_one' := by { ... },
   map_mul' := by { ... } }

lemma selmer.val_ker : selmer.val.ker = K(∅, n).subgroup_of K(S, n)
```

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```
def f : units (O K) →* K(∅, n) :=
 { to_fun := λ x, ⟨quotient_group.mk $ ne_zero_of_unit x, λ p _, val_of_unit_mod p x⟩,
   map_one' := rfl,
   map_mul' := λ ⟨⟨_, _⟩, ⟨_, _⟩, _, _⟩ ⟨⟨_, _⟩, ⟨_, _⟩, _, _⟩, rfl } -- lol

lemma f_ker : f.ker = (units (O K))^n

def g : K(∅, n) →* class_group (O K) K := ... -- hmm

lemma g_ker : g.ker = f.range
```

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Note the classical n-Selmer group of E is

$$\mathrm{Sel}(K, E[n]) \leq K(S,n) \times K(S,n).$$

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Ongoing for $K = \mathbb{Q}$. Probably not ready for general K?

Future

Potential future projects:

- n-division polynomials and structure of $E(K)[n]$
- formal groups and local theory
- ramification theory $\implies$ full Mordell-Weil theorem
- Galois cohomology $\implies$ Selmer and Tate-Shafarevich groups
- modular functions $\implies$ complex theory
- algebraic geometry $\implies$ associativity, finally

Thank you!