Mathematical Theorem Proving Workshop

Monday, 25 April 2022 — Cambridge Research Centre

Elliptic Curves in Lean

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Informally

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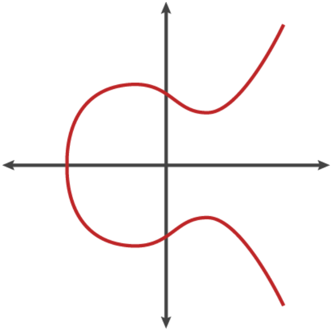

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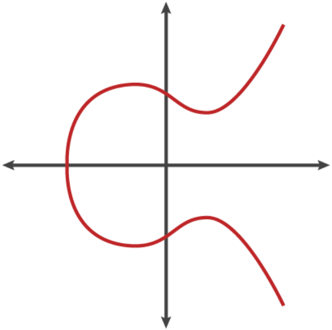

- Points form a group!

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- Distribution of ranks of rational elliptic curves
  The BSD conjecture (analytic rank equals algebraic rank)

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Good for algebraic geometry, but not very friendly...

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Group law is free, but still need equations...

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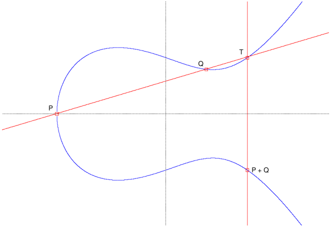
The *group law* is reduced to drawing lines.



Implementation

Three definitions of elliptic curves:

1. Abstract definition over a scheme
2. Abstract definition over a field
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Generality: $1 \supset 2 \overset{\text{RR}}{=} 3$

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 -(a₁^2 + 4*a₂)^2*(a₁^2*a₆ + 4*a₂*a₆ - a₁*a₃*a₄ + a₂*a₃^2 - a₄^2)
 - 8*(2*a₄ + a₁*a₃)^3 - 27*(a₃^2 + 4*a₆)^2
 + 9*(a₁^2 + 4*a₂)*(2*a₄ + a₁*a₃)*(a₃^2 + 4*a₆)

structure EllipticCurve (R : Type) [comm_ring R] :=
 (a₁ a₂ a₃ a₄ a₆ : R) (disc : units R) (disc_eq : disc.val = disc_aux a₁ a₂ a₃ a₄ a₆)
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This is the *scheme* E, but what about the *abelian group* $E(F)$?

Points

```
variables {F : Type} [field F] (E : EllipticCurve F) (K : Type) [field K] [algebra F K]

inductive point
  | zero
  | some (x y : K) (w : y^2 + E.a₁*x*y + E.a₃*y = x^3 + E.a₂*x^2 + E.a₄*x + E.a₆)

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Negation

```
def neg : E(K) → E(K)
  | zero := zero
  | (some x y w) := some x (−y − E.a₁*x − E.a₃) $ by { rw [← w], ring }

instance : has_neg E(K) := ⟨neg⟩
```

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Addition

```
def add : E(K) → E(K) → E(K)
  | zero P := P
  | P zero := P
  | (some x₁ y₁ w₁) (some x₂ y₂ w₂) :=
    if x_ne : x₁ ≠ x₂ then
     let L := (y₁ − y₂) / (x₁ − x₂),
         x₃ := L^2 + E.a₁*L − E.a₂ − x₁ − x₂,
         y₃ := −L*x₃ − E.a₁*x₃ − y₁ + L*x₁ − E.a₃
     in some x₃ y₃ $ by { ... }
    else if y_ne : y₁ + y₂ + E.a₁*x₂ + E.a₃ ≠ 0 then
     ...
    else
     zero

instance : has_add E(K) := ⟨add⟩
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Commutativity is doable

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lemma add_comm (P Q : E(K)) : P + Q = Q + P :=
 by { rcases ⟨P, Q⟩ with ⟨_ | _, _ | _⟩, ... }
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Associativity is difficult

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lemma add_assoc (P Q R : E(K)) : (P + Q) + R = P + (Q + R) :=
 by { rcases ⟨P, Q, R⟩ with ⟨_ | _, _ | _, _ | _⟩, . . . }
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- Proved (by E-I Bartzia and P-Y Strub) in Coq [4]
  that $E(K) \cong \mathrm{Pic}^0_{E/F}(K)$ for $\mathrm{char}\ F \neq 2, 3$

[4] A Formal Library for Elliptic Curves in the Coq Proof Assistant (2015)

Progress

Modulo associativity, what has been done?

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Functoriality $\mathbf{Alg}_F \to \mathbf{Ab}$

```
def point_hom (φ : K →ₐ[F] L) : E(K) → E(L)
  | zero := zero
  | (some x y w) := some (φ x) (φ y) $ by { . . . }

local notation K →[F] L := (algebra.of_id K L).restrict_scalars F

lemma point_hom.id (P : E(K)) : point_hom (K →[F] K) P = P := by cases P; refl

lemma point_hom.comp (P : E(K)) :
  point_hom (L →[F] M) (point_hom (K →[F] L) P) = point_hom ((L →[F] M).comp (K →[F] L)) P := by cases P; refl
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Galois module structure $\mathrm{Gal}(L/K) \curvearrowright E(L)$

```
def point_gal (σ : L ≃ₐ[K] L) : E(L) → E(L)
  | zero := zero
  | (some x y w) := some (σ • x) (σ • y) $ by { . . . }

lemma point_gal.fixed : mul_action.fixed_points (L ≃ₐ[K] L) E(L) = (point_hom (K →[F] L)).range := by { . . . }
```

Progress

Modulo associativity, what has been done?

Isomorphisms $(x, y) \mapsto (u^2x + r, u^3y + u^2sx + t)$

```
variables (u : units F) (r s t : F)

def cov : EllipticCurve F :=
{ a1 := u.inv*(E.a1 + 2*s),
  a2 := u.inv^2*(E.a2 - s*E.a1 + 3*r - s^2),
  a3 := u.inv^3*(E.a3 + r*E.a1 + 2*t),
  a4 := u.inv^4*(E.a4 - s*E.a3 + 2*r*E.a2 - (t + r*s)*E.a1 + 3*r^2 - 2*s*t),
  a6 := u.inv^6*(E.a6 + r*E.a4 + r^2*E.a2 + r^3 - t*E.a3 - t^2 - r*t*E.a1),
  disc := ⟨u.inv^12*E.disc.val, u.val^12*E.disc.inv, by { . . . }, by { . . . }⟩,
  disc_eq := by { simp only, rw [disc_eq, disc_aux, disc_aux], ring } }

def cov.to_fun : (E.cov u r s t)(K) → E(K)
  | zero := zero
  | (some x y w) := some (u.val^2*x + r) (u.val^3*y + u.val^2*s*x + t) $ by { . . . }

def cov.inv_fun : E(K) → (E.cov u r s t)(K)
  | zero := zero
  | (some x y w) := some (u.inv^2*(x - r)) (u.inv^3*(y - s*x + r*s - t)) $ by { . . . }

def cov.equiv_add : (E.cov u r s t)(K) ≃+ E(K) :=
  ⟨cov.to_fun u r s t, cov.inv_fun u r s t, by { . . . }, by { . . . }, by { . . . }⟩
```

Progress

Modulo associativity, what has been done?

2-division polynomial $\psi_2(x)$

```
def ψ₂_x : cubic K := ⟨4, E.a₁^2 + 4*E.a₂, 4*E.a₄ + 2*E.a₁*E.a₃, E.a₃^2 + 4*E.a₆⟩

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Structure of $E(K)[2]$

```
notation E(K)[n] := ((·) n : E(K) →+ E(K)).ker

lemma E₂.x {x y w} : some x y w ∈ E(K)[2] ↔ x ∈ (ψ₂_x E K).roots := by { . . . }

theorem E₂.card_le_four : fintype.card E(K)[2] ≤ 4 := by { . . . }

variables [algebra ((ψ₂_x E F).splitting_field) K]

theorem E₂.card_eq_four : fintype.card E(K)[2] = 4 := by { . . . }

lemma E₂.gal_fixed (σ : L ≃ₐ[K] L) (P : E(L)[2]) : σ · P = P := by { . . . }
```

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Soon: Mordell's theorem for $E[2] \subset E(\mathbb{Q})$.

Future

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- n-division polynomials and the structure of $E(K)[n]$
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- ramification theory $\implies$ Mordell-Weil theorem for number fields
- Galois cohomology $\implies$ Selmer and Tate-Shafarevich groups
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Thank you!