

Young Researchers in Algebraic Number Theory

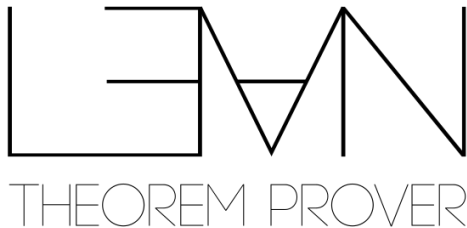
Wednesday, 24 August 2022

# Formalisation of elliptic curves in Lean

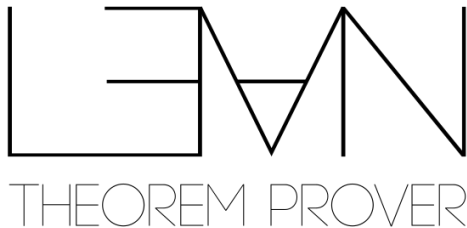
David Kurniadi Angdinata

London School of Geometry and Number Theory

# The Lean theorem prover

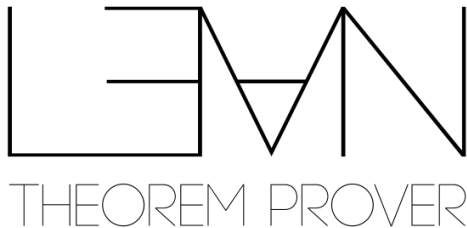


# The Lean theorem prover



A functional programming language...

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A functional programming language...

and an interactive theorem prover!

# Programming in Lean

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Can define inductive types.

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inductive Nat
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| succ : Nat → Nat
```

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$$\text{element} \in \text{set} \implies \text{Term} : \text{Type}$$

Can define inductive types.

```
inductive Nat
| zero : Nat
| succ : Nat → Nat
```

Can define functions recursively.

```
def add : Nat → Nat → Nat
| n zero := n
| n (succ m) := succ (add n m)
```

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begin
  intro n,
  induction n with m hm,
  { refl },
  { rw [add, hm] }
end
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The keywords `intro`, `induction`, `refl`, and `rw` are **tactics**.

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Play **The Natural Number Game!**

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3k files, 1m lines, 40k definitions, 100k theorems, 270 contributors.

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Consider the following theorem in `group_theory/quotient_group`.

```
variables {G H : Type} [group G] [group H]
variables (φ : G →* H) (ψ : H → G) (hφ : right_inverse ψ φ)

def quotient_ker_equiv_of_right_inverse : G / ker φ ≃* H :=
{ to_fun := ker_lift φ,
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Consider an immediate corollary.

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Canonical isomorphisms are important data!

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Here is a working definition in `algebraic_geometry/EllipticCurve`.

```
def Δ_aux {R : Type} [comm_ring R] (a1 a2 a3 a4 a6 : R) : R :=
  let
    b2 := a1^2 + 4*a2,
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    b6 := a3^2 + 4*a6,
    b8 := a1^2*a6 + 4*a2*a6 - a1*a3*a4 + a2*a3^2 - a4^2
  in
    -b2^2*b8 - 8*b4^3 - 27*b6^2 + 9*b2*b4*b6

structure EllipticCurve (R : Type) [comm_ring R] :=
  (a1 a2 a3 a4 a6 : R) (Δ : units R) (Δ_eq : ↑Δ = Δ_aux a1 a2 a3 a4 a6)
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Much can be done just with this definition.

# Elliptic curves in Lean

Can define  $K$ -points.

```
variables {F : Type} [field F] (E : EllipticCurve F) (K : Type) [field K] [algebra F K]

inductive point
| zero
| some (x y : K) (w : y^2 + E.a1*x*y + E.a3*y = x^3 + E.a2*x^2 + E.a4*x + E.a6)

notation E(K) := point E K
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Can define negation.

```
def neg : E(K) → E(K)
| zero := zero
| (some x y w) := some x (-y - E.a1*x - E.a3)
begin
  rw [← w],
  ring
end

instance : has_neg E(K) := ⟨neg⟩
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Can define addition.

```
def add : E(K) → E(K) → E(K)
| zero P := P
| P zero := P
| (some x1 y1 w1) (some x2 y2 w2) :=
  if x_ne : x1 ≠ x2 then
    let
      L := (y1 - y2) / (x1 - x2),
      x3 := L^2 + E.a1*L - E.a2 - x1 - x2,
      y3 := -L*x3 - E.a1*x3 - y1 + L*x1 - E.a3
    in
      some x3 y3 ... -- 100 lines
  else ... -- 100 lines

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Can prove group axioms

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lemma zero_add (P : E(K)) : 0 + P = P := ...

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lemma add_comm (P Q : E(K)) : P + Q = Q + P := ... -- 100 lines

lemma add_assoc (P Q R : E(K)) : (P + Q) + R = P + (Q + R) := ... -- ?? lines
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Can also prove Galois-theoretic properties and structure of torsion points.

# The Mordell-Weil theorem in Lean

Can prove Mordell's theorem by complete 2-descent and naïve heights.

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- ▶ Reduce to  $K \supseteq E[2]$ , so that  $y^2 = (x - e_1)(x - e_2)(x - e_3)$ .
- ▶ Define the complete 2-descent homomorphism

$$\begin{array}{ccc} E(K) & \longrightarrow & K^\times / (K^\times)^2 \times K^\times / (K^\times)^2 \\ \mathcal{O} & \longmapsto & (1, 1) \\ (x, y) & \longmapsto & (x - e_1, x - e_2) \end{array} .$$

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- ▶ Prove  $\text{Cl}_K$  is finite (done) and  $\mathcal{O}_K^\times$  is finitely generated (soon).  $\square$



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$$\begin{aligned} h : E(\mathbb{Q}) &\longrightarrow \mathbb{R} \\ \mathcal{O} &\longmapsto 0 \\ \left(\frac{n}{d}, y\right) &\longmapsto \log \max(|n|, |d|) \end{aligned} .$$

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- ▶ Prove  $\forall C \in \mathbb{R}$ , the set  $\{P \in E(\mathbb{Q}) \mid h(P) \leq C\}$  is finite.

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- ▶ Prove  $\forall Q \in E(\mathbb{Q}), \exists C \in \mathbb{R}, \forall P \in E(\mathbb{Q}), h(P + Q) \leq 2h(P) + C$ .
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# The Mordell-Weil theorem in Lean

Can prove Mordell's theorem by complete 2-descent and naïve heights.

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Can finally define the algebraic rank of  $E(\mathbb{Q})$ .

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# Thank you!

Check out the leanprover community!