Ideal Class Groups [1]

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LSGNT

Short introductory talk

Wednesday, 6 October 2021

[1] of number fields

Diophantine equations!

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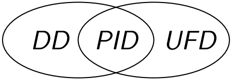

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Fact

$$\mathrm{Nm}(I \cdot J) = \mathrm{Nm}(I)\mathrm{Nm}(J), \qquad \mathrm{Nm}(\langle x\rangle) = \mathrm{Nm}(x) = \prod_{\sigma : K \to \overline{K}} \sigma(x).$$

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Every $[I] \in \mathrm{Cl}(K)$ has a representative $I \trianglelefteq \mathcal{O}_K$ with $\mathrm{Nm}(I) \leq \mathrm{M}_K$. $\square$

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