Sheaves, functors, and derived versions

Study group on character sheaves

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Tuesday, 4 November 2025

Presheaves

Throughout, let R be a ring, and let X, Y, and Z be topological spaces. Then U and U_i (resp V and V_i) will be open sets of X (resp Y), and $\mathcal{F}$ and $\mathcal{F}_i$ (resp $\mathcal{G}$ and $\mathcal{G}_i$) will be sheaves of R-modules on X (resp Y).

A **presheaf** (of R-modules on X) is a functor $\mathcal{F} : \mathbf{Top}(X)^{\mathrm{op}} \to \mathbf{Mod}_R$. In other words, it associates every $U \in \mathbf{Top}(X)$ to some $\mathcal{F}(U) \in \mathbf{Mod}_R$, and for all $U_1, U_2 \in \mathbf{Top}(X)$ with $U_1 \subseteq U_2$, there are restrictions

$$(-)|_{U_1}^{U_2} : \mathcal{F}(U_1 \to U_2) : \mathcal{F}(U_2) \to \mathcal{F}(U_1),$$

such that

- $(-)|_{U_1}^{U_1} = \mathrm{id}$, and
- $((-)|_{U_1}^{U_2})|_{U_2}^{U_3} = (-)|_{U_1}^{U_3}$ for all $U_3 \in \mathbf{Top}(X)$ with $U_2 \subseteq U_3$.

Let $\mathbf{PSh}(X, R)$ denote the category of presheaves (of R-modules on X).

Sheaves

A **sheaf** (of R-modules on X) is a presheaf $\mathcal{F} \in \mathbf{PSh}(X, R)$ such that, if $\{U_i\}_i$ is an open cover of $U \in \mathbf{Top}(X)$, then the (equaliser) sequence

$$0 \to \mathcal{F}(U) \xrightarrow{s \mapsto (s|^U_{U_i})_i} \prod_i \mathcal{F}(U_i) \underset{(s_j \mapsto (s_j|^{U_j}_{U_i \cap U_j})_i)_j}{\overset{(s_i \mapsto (s_i|^{U_i}_{U_i \cap U_j})_j)_i}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact. In other words,

S1 if $s \in \mathcal{F}(U)$ is such that $s|^U_{U_i} = 0$ for all i, then $s = 0$, and

S2 if $s_i \in \mathcal{F}(U_i)$ and $s_j \in \mathcal{F}(U_j)$ are such that $s_i|^{U_i}_{U_i \cap U_j} = s_j|^{U_j}_{U_i \cap U_j}$ for all i and j, then there is some $s \in \mathcal{F}(U)$ such that $s|^U_{U_i} = s_i$ for all i.

Let $\mathbf{Sh}(X, R)$ denote the category of sheaves (of R-modules on X), and let $(-)^- : \mathbf{Sh}(X, R) \to \mathbf{PSh}(X, R)$ denote its natural forgetful functor.

Morphisms of sheaves

A **morphism** of (pre)sheaves (of R-modules on X) is a natural transformation $\phi : \mathcal{F}_1 \to \mathcal{F}_2$. In other words, it is a collection of R-linear maps $\phi_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ for each $U \in \mathbf{Top}(X)$, such that

$$\begin{array}{ccc} \mathcal{F}_1(U_1) & \xrightarrow{\phi_{U_1}} & \mathcal{F}_2(U_1) \\ {\scriptstyle (-)|^{U_1}_{U_2}}\big\downarrow & & \big\downarrow{\scriptstyle (-)|^{U_1}_{U_2}} \\ \mathcal{F}_1(U_2) & \xrightarrow[\phi_{U_2}]{} & \mathcal{F}_2(U_2) \end{array} .$$

The **stalk** of $\mathcal{F}$ at some $x \in X$ is the direct limit

$$\mathcal{F}_x := \varinjlim_{\substack{U \in \mathbf{Top}(X), \\ x \in U}} \mathcal{F}(U).$$

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are sheaves, then ϕ is an isomorphism precisely if the induced morphism $\phi_x : \mathcal{F}_{1,x} \to \mathcal{F}_{2,x}$ is an isomorphism for each $x \in X$.

Examples of sheaves

Let X be a C^n-manifold over $K/\mathbb{R}$. For all $m \leq n$, there are sheaves

$$U \mapsto C^m(U, K).$$

Let X be a variety over $K = \overline{K}$. The **structure sheaf** is given by

$$\mathcal{O}_X : U \mapsto \{\text{regular functions } U \to K\}.$$

Let M be an R-module, and let $x \in X$. The **skyscraper sheaf** is given by

$$\underline{M_x} : U \mapsto \begin{cases} M & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, the presheaf

$$\mathcal{F} : U \mapsto \{\text{bounded continuous functions } U \to \mathbb{R}\}$$

is not a sheaf.

Constant sheaves

Let M be an R-module. The constant sheaf $\underline{M_X}$ is not just the presheaf $U \mapsto M$! Since $\emptyset$ has an empty open cover $\{U_i\}_{i\in\emptyset}$, all $s \in \underline{M_X}(\emptyset)$ vacuously satisfy $s|^{\emptyset}_{U_i} = 0$ for all $i \in \emptyset$, so S1 says that $s = 0$. Thus

$$\underline{M_X}(\emptyset) = 0.$$

Let $U_1, U_2 \in \mathbf{Top}(X)$ be disjoint with $\underline{M_X}(U_1) = \underline{M_X}(U_2) = M$. If $s_1 \in \underline{M_X}(U_1)$ and $s_2 \in \underline{M_X}(U_2)$, then $s_1|^{U_1}_{U_1 \sqcap U_2} = s_2|^{U_2}_{U_1 \sqcap U_2} = 0$, so S2 gives some $s \in \underline{M_X}(U_1 \sqcup U_2)$ such that $s|^{U_1 \sqcup U_2}_{U_1} = s_1$ and $s|^{U_1 \sqcup U_2}_{U_2} = s_2$. Thus

$$\underline{M_X}(U_1 \sqcup U_2) = M \oplus M.$$

In other words, the **constant sheaf** is given by

$$\underline{M_X} : U \mapsto \{\text{continuous functions } U \to M\},$$

where M is given the discrete topology.

Sheafification

Let $\mathcal{F} \in \mathbf{PSh}(X,R)$. The **sheafification** of $\mathcal{F}$ is the unique sheaf $\mathcal{F}^+ \in \mathbf{Sh}(X,R)$ satisfying the universal property

$$\begin{tikzcd} \mathcal{F} \arrow[r, "(-)^+"] \arrow[dr, "\forall\phi"'] & \mathcal{F}^+ \arrow[d, dashed, "\exists!\phi^+"] \\ & \forall\mathcal{F}_0 \end{tikzcd} .$$

This says that for any $\mathcal{F}_0 \in \mathbf{Sh}(X,R)$ and any $\phi : \mathcal{F} \to \mathcal{F}_0$, there is a unique $\phi^+ : \mathcal{F}^+ \to \mathcal{F}_0$ such that $\phi^+ \circ (-)^+ = \phi$.

In other words, $(-)^+ : \mathbf{PSh}(X,R) \to \mathbf{Sh}(X,R)$ is the **right adjoint** to the forgetful functor $(-)^- : \mathbf{Sh}(X,R) \to \mathbf{PSh}(X,R)$, in the sense that

$$\mathrm{Hom}_{\mathbf{Sh}(X,R)}(\mathcal{F}_1^+, \mathcal{F}_2) \cong \mathrm{Hom}_{\mathbf{PSh}(X,R)}(\mathcal{F}_1, \mathcal{F}_2^-),$$

so that $\mathcal{F}_x = \mathcal{F}_x^+$ for all $x \in X$.

Hom and tensor product

Grothendieck introduced a six-functor formalism for sheaves.

The **hom** $\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2) \in \mathbf{Sh}(X, R)$ is the sheaf

$$U \mapsto \mathrm{Hom}_{\mathbf{Sh}(U,R)}(\mathcal{F}_1|_U, \mathcal{F}_2|_U).$$

The **tensor product** $\mathcal{F}_1 \otimes \mathcal{F}_2 \in \mathbf{Sh}(X, R)$ is the sheafification of

$$U \mapsto \mathcal{F}_1(U) \otimes_R \mathcal{F}_2(U).$$

Fact

- $\mathrm{Hom}_{\mathbf{Sh}(X,R)}(\mathcal{F}_1 \otimes \mathcal{F}_2, \mathcal{F}_3) \cong \mathrm{Hom}_{\mathbf{Sh}(X,R)}(\mathcal{F}_1, \mathcal{H}om(\mathcal{F}_2, \mathcal{F}_3))$.
- $\mathcal{F} \otimes \underline{R_X} \cong \mathcal{F}$ *and* $\mathcal{H}om(\underline{R_X}, \mathcal{F}) \cong \mathcal{F}$.
- *If* $x \in X$, *then* $(\mathcal{F}_1 \otimes \mathcal{F}_2)_x \cong \mathcal{F}_{1,x} \otimes_R \mathcal{F}_{2,x}$, *but* $\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2)_x \ncong \mathrm{Hom}(\mathcal{F}_{1,x}, \mathcal{F}_{2,x})$ *in general.*

Pullback and pushforward

Let $f : X \to Y$. The **pushforward** $f_*\mathcal{F} \in \mathbf{Sh}(Y, R)$ is the sheaf

$$V \mapsto \mathcal{F}(f^{-1}(V)).$$

The **pullback** $f^*\mathcal{G} \in \mathbf{Sh}(X, R)$ is the sheafification of

$$U \mapsto \varinjlim_{\substack{V \in \mathbf{Top}(Y), \\ f(U) \subseteq V}} \mathcal{G}(V).$$

Fact

- $\mathrm{Hom}_{\mathbf{Sh}(X,R)}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Sh}(Y,R)}(\mathcal{G}, f_*\mathcal{F})$.
- $f^*\underline{R_Y} = \underline{R_X}$ *and* $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)}$ *for all* $x \in X$.
- *If* $\iota_y : \{y\} \hookrightarrow Y$ *for some* $y \in Y$*, then* $\iota_y^*\mathcal{G} = \underline{\mathcal{G}_{y,\{y\}}}$.
- *If* $\pi^x : X \twoheadrightarrow \{x\}$ *for some* $x \in X$*, then* $\pi_*^x\mathcal{F} = \underline{\mathcal{F}(X)}$.
- *If* $g : Y \to Z$*, then* $(g \circ f)_* = g_* \circ f_*$ *and* $(g \circ f)^* = f^* \circ g^*$.

Shriek pushforward

Recall that f is **proper** if it is universally closed, in the sense that $f \times \mathrm{id} : X \times Z \to Y \times Z$ is closed for all Z. If X is locally compact Hausdorff, then f is proper iff $f^{-1}(Z)$ is compact for any compact $Z \subseteq Y$. The **shriek pushforward** $f_! \mathcal{F} \in \mathbf{Sh}(Y, R)$ is the sheaf

$$V \mapsto \{s \in \mathcal{F}(f^{-1}(V)) : f|_{\mathrm{supp}(s)} \text{ is proper}\},$$

where $\mathrm{supp}(s) := \{x \in X : s \neq 0 \text{ in } \mathcal{F}_x\}$ is closed.

Fact

- *If $\iota : X \hookrightarrow Y$ is open, then* $\mathrm{Hom}_{\mathbf{Sh}(Y,R)}(\iota_! \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathbf{Sh}(X,R)}(\mathcal{F}, \iota^* \mathcal{G})$.
- *If f is proper, such as when $f : X \hookrightarrow Y$ is closed, then $f_! = f_*$.*
- *If $\pi^x : X \twoheadrightarrow \{x\}$ for some $x \in X$, then*

$$\pi^x_! \mathcal{F} = \underline{\{s \in \mathcal{F}(X) : \mathrm{supp}(s) \textit{ is compact}\}}.$$

- *If $g : Y \to Z$ is separated, in the sense that the diagonal $Y \hookrightarrow Y \times_Z Y$ is closed, then $(g \circ f)_! = g_! \circ f_!$.*

Locally closed inclusions

Assume that $\iota : X \hookrightarrow Y$ is locally closed. Then $\iota_! : \mathbf{Sh}(X, R) \to \mathbf{Sh}(Y, R)$ is **extension-by-zero**, where $\iota_! \mathcal{F} \in \mathbf{Sh}(Y, R)$ is the sheafification of

$$V \mapsto \begin{cases} \mathcal{F}(V \cap \iota(X)) & \text{if } V \cap \overline{\iota(X)} \subseteq \iota(X), \\ 0 & \text{otherwise,} \end{cases}$$

so its stalk at $y \in Y$ is

$$(\iota_! \mathcal{F})_y = \begin{cases} \mathcal{F}_y & \text{if } y \in \iota(X), \\ 0 & \text{otherwise.} \end{cases}$$

In this case, $\iota_!$ has a right adjoint **restriction-with-supports** $\iota^! : \mathbf{Sh}(Y, R) \to \mathbf{Sh}(X, R)$, where $\iota^! \mathcal{G} \in \mathbf{Sh}(X, R)$ is the sheafification of

$$U \mapsto \varinjlim_{\substack{V \in \mathbf{Top}(Y), \\ V \cap \overline{\iota(X)} = \iota(U)}} \{ s \in \mathcal{G}(V) : \operatorname{supp}(s) \subseteq \iota(U) \},$$

so that $\iota^! = \iota^*$ whenever ι is open.

Classical derived functors

Since $\mathbf{Mod}_R$ has enough injectives, $\mathbf{Sh}(X, R)$ also has enough injectives, so for any $\mathcal{F} \in \mathbf{Sh}(X, R)$, there is a **classical injective resolution**

$$0 \to \mathcal{F} \to \mathcal{I}^0 \xrightarrow{d^0} \mathcal{I}^1 \xrightarrow{d^1} \dots .$$

Let $F : \mathbf{Sh}(X, R) \to \mathbf{Sh}(Y, R)$ be a functor. For each $i \in \mathbb{N}$, the **classical derived functor** $R^i F : \mathbf{Sh}(X, R) \to \mathbf{Sh}(Y, R)$ of F is given by

$$\mathcal{F} \mapsto H^i(0 \to F(\mathcal{I}^0) \xrightarrow{F(d^0)} F(\mathcal{I}^1) \xrightarrow{F(d^1)} \dots) := \ker F(d^i) / \operatorname{im} F(d^{i-1}),$$

which is independent of the choice of classical injective resolution. For each $i \in \mathbb{Z}$, the **cohomology** of $\mathcal{F}$ is

$$H^i(\mathcal{F}) := R^i F(\mathcal{F}).$$

If F is left exact, then $H^0(\mathcal{F}) = R^0 F(\mathcal{F}) = \ker F(d^0) = F(\mathcal{F})$. For instance, $\mathcal{H}om(\mathcal{F}, -)$, $\mathcal{H}om(-, \mathcal{F})$, f^*, f_*, $f_!$, $\iota_!$, and $\iota^!$ are all left exact, and f^* and $\iota_!$ (and $\mathcal{F} \otimes -$ and $- \otimes \mathcal{F}$ if $\mathbf{Mod}_R$ is flat) are also right exact.

Complex category

Let $\mathcal{A}$ be an abelian category. Let $C(\mathcal{A})$ denote the category whose objects are **chain complexes** $A^\bullet$ for some $A^i \in \mathcal{A}$ given by

$$\dots \xrightarrow{d_A^{i-1}} A^i \xrightarrow{d_A^i} A^{i+1} \xrightarrow{d_A^{i+1}} \dots,$$

and whose morphisms are **chain maps** $\phi^\bullet : A^\bullet \to B^\bullet$ such that

$$\begin{tikzcd}
\dots \arrow[r, "d_A^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "\phi^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "\phi^{i+1}"] & \dots \\
\dots \arrow[r, "d_B^{i-1}"'] & B^i \arrow[r, "d_B^i"'] & B^{i+1} \arrow[r, "d_B^{i+1}"'] & \dots
\end{tikzcd}.$$

For each $i \in \mathbb{Z}$, the **cohomology** of a chain complex $A^\bullet \in \mathcal{A}$ is given by

$$H^i(A^\bullet) := \ker d^i / \operatorname{im} d^{i-1}.$$

A chain map $\phi^\bullet : A^\bullet \to B^\bullet$ is a **quasi-isomorphism** if the induced morphisms $H^i(\phi^\bullet) : H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms for all $i \in \mathbb{Z}$.

Derived category

Let $\mathcal{C}$ be a category. The **localisation** of $\mathcal{C}$ with respect to a collection S of morphisms is a category $S^{-1}\mathcal{C}$ satisfying the universal property

$$\begin{tikzcd} \mathcal{C} \arrow[r, "S^{-1}"] \arrow[dr, "\forall F"'] & S^{-1}\mathcal{C} \arrow[d, "\exists ! S^{-1}F"] \\ & \forall \mathcal{C}_0 \end{tikzcd},$$

where $\mathcal{C}_0$ is any category such that $F(\phi)$ is an isomorphism for all $\phi \in S$.

The **derived category** $D(\mathcal{A})$ of $\mathcal{A}$ is the localisation of $C(\mathcal{A})$ with respect to quasi-isomorphisms. Furthermore, let $D^+(\mathcal{A})$ and $D^-(\mathcal{A})$ denote its subcategories such that $A^i = 0$ for sufficiently large or small $i \in \mathbb{Z}$ respectively, and let $D^b(\mathcal{A}) := D^+(\mathcal{A}) \cap D^-(\mathcal{A})$.

Similarly, let $C^*(\mathcal{A})$ denote the same for $C(\mathcal{A})$ for each of $* \in \{+, -, b\}$.

Derived functors

Assume that $\mathcal{A}$ has enough injectives. Then for all $A^\bullet \in C(\mathcal{A})$, there is an **injective resolution** $I^\bullet \in C(\mathcal{A})$ with a quasi-isomorphism

$$A^\bullet \to I^\bullet.$$

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. By abstract nonsense, it preserves quasi-isomorphisms on $C^+(\mathcal{A})$, so it defines a functor $F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$. Furthermore, there is a **derived functor** $RF : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ given by

$$A^\bullet \mapsto F(I^\bullet),$$

which recovers the classical derived functor for each $i \in \mathbb{Z}$ by

$$R^i F(A) = H^i(RF(A)).$$

If it is also right exact, then it preserves quasi-isomorphisms on $C^-(\mathcal{A})$, so it defines a functor $F : D(\mathcal{A}) \to D(\mathcal{B})$, and the derived functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ satisfies $RF(A^\bullet) = 0$ for all $A^\bullet \in \mathcal{A}$.

Derived sheaf functors

Let $D^*(X,R) := D^*(\mathbf{Sh}(X,R))$, which has non-zero derived functors

$$R\mathcal{H}om(\mathcal{F},-),\ R\mathcal{H}om(-,\mathcal{F}),\ Rf_*,\ Rf_!,\ \iota^!.$$

The **shriek pullback** $f^! : D^+(Y,R) \to D^+(X,R)$ is the right adjoint of $Rf_! : D^+(X,R) \to D^+(Y,R)$, which exists when X and Y are locally compact Hausdorff. If $\iota : X \hookrightarrow Y$ is locally closed, then this coincides with $R\iota^! : D^+(Y,R) \to D^+(X,R)$.

Fact

- *If $\pi^x : X \to \{x\}$ for some $x \in X$, then $R^i\pi^x_*\mathcal{F} = H^i(\mathcal{F})$ and $R^i\pi^x_!\mathcal{F} = H^i_c(\mathcal{F})$.*
- *If $f : X \to Y$ and $g : Y \to Z$, and X, Y, and Z are locally compact Hausdorff, then $(Rg \circ Rf)_* = Rg_* \circ Rf_*$ and $(Rg \circ Rf)_! = Rg_! \circ Rf_!$.*
- *Proper base change: if $f : X \to Y$ and $h : Z \to X$, and $\pi_X : X \times_Y Z \to X$ and $\pi_Z : X \times_Y Z \to Z$, then $h^* \circ Rf_! \cong R\pi_{Z!} \circ \pi_Z^*$.*