Schinzel's hypothesis H

David Ang

Open problems in number theory

Thursday, 31 October 2024

Some fun quotes

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Bunyakovsky (1857):

Il est à présumer que la démonstration rigoureuse du théorème énoncé sur les progressions arithmétiques des ordres supérieurs conduirait, dans l'état actuel de la théorie des nombres, à des difficultés insurmontables; néanmoins, sa réalité ne peut pas être révoquée en doute.

Primes in arithmetic progressions

Theorem (Dirichlet, 1837)

Let $a, b \in \mathbb{Z}$*. Assume no primes* p *satisfy* $p \mid a$ *and* $p \mid b$*. Then there are infinitely many* n *such that* $an + b$ *is prime.*

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Example ($4X + 3$)

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$4n+3$	3	7	11	15	19	23	27	31	35	39	43	47	51
prime	✓	✓	✓		✓	✓		✓			✓	✓	

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If there were a finite set $S := \{p \text{ prime} : p \equiv 3 \mod 4\}$, then

$$N := 2 + \prod_{p \in S} p^2 \equiv 3 \mod 4,$$

so N has a prime factor $q \equiv 3 \mod 4$ not in S, which is a contradiction.

Primes in polynomial sequences

Conjecture (Bunyakovsky, 1857)

Let $f \in \mathbb{Z}[X]$ be irreducible. Assume no primes p satisfy "$p \mid f(n)$ for all n". Then there are infinitely many n such that $f(n)$ is prime.

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n	0	1	2	3	4	5	6	7	8	9	10	11	12
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This is one of the four Landau's problems, amongst Goldbach's conjecture, the twin prime conjecture, and Legendre's conjecture.

Simultaneous primes in arithmetic progressions

Conjecture (Dickson, 1904)

Let $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Z}$. *Set* $f(X) := (a_1 X + b_1) \cdot \cdots \cdot (a_k X + b_k)$. *Assume no primes* p *satisfy "*$p \mid f(n)$ *for all* n*". Then there are infinitely many* n *such that* $a_1 n + b_1, \ldots, a_k n + b_k$ *are simultaneously prime.*

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Example (X and $2X + 1$)

p	2	3	5	7	11	13	17	19	23	29	31	37	41
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This is the Germain prime conjecture, which implies that there are infinitely many composite Mersenne numbers, since $2p + 1 \mid 2^p - 1$ whenever $p \equiv 3 \mod 4$ is a Germain prime.

Density of simultaneous primes

Conjecture (Hardy–Littlewood, 1923)

Let $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{Z}$*. Set* $f(X) := (a_1 X + b_1) \cdot \cdots \cdot (a_k X + b_k)$*. Assume no primes* p *satisfy "*$p \mid f(n)$ *for all* n*". Then*

$$\#\left\{ n \leq N : \begin{array}{c} a_1 n + b_1, \ldots, a_k n + b_k \\ \text{are simultaneously prime} \end{array} \right\} \sim C \cdot \frac{N}{\log^k N}.$$

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Here,

$$C := \prod_p \left(1 - \frac{1}{p}\right)^{-k} \left(1 - \frac{\#\{n \in \mathbb{F}_p : f(n) = 0\}}{p}\right).$$

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If $f_1(X) = X$ and $f_2(X) = X + 2$, then C is the twin prime constant.

Simultaneous primes in polynomial sequences

Conjecture (Schinzel's hypothesis H, 1958)

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Conjecture (Bateman–Horn, 1962)

Let $f_1, \ldots, f_k \in \mathbb{Z}[X]$ be irreducible. Set $f := f_1 \cdot \cdots \cdot f_k$. Assume no primes p satisfy "$p \mid f(n)$ for all n". Then

$$\#\left\{n \leq N : \begin{array}{c} f_1(n), \ldots, f_k(n) \\ \text{are simultaneously prime} \end{array}\right\} \sim C \cdot \frac{N}{\prod_i \deg f_i \cdot \log^k N}.$$

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Multivariate variants

Theorem (Friedlander–Iwaniec, 1997)

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Let $f_1, \ldots, f_k \in \mathbb{Z}[X]$ *such that* $f_i(0) = 0$*. Then there are infinitely many* $(x, y) \in \mathbb{Z}^2$ *such that* $x + f_1(y), \ldots, x + f_k(y)$ *are simultaneously prime.*

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Theorem (Bodin–Dèbes–Najib, 2019)

Let R *be a characteristic zero UFD whose fraction field satisfies the product formula, and let* $f_1, \ldots, f_k \in R[X, Y]$. *Then there are* $y \in R[X]$ *such that* $f_1(X, y(X)), \ldots, f_k(X, y(X))$ *are simultaneously irreducible.*

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Example ($X^8 + t^3$ over $\mathbb{F}_2[t]$)

$(t^2+t+1)^8+t^3 = (t+1)(t^{15}+t^{14}+t^{13}+t^{12}+t^{11}+t^{10}+t^9+t^8+t^2+t+1).$

Genericity of simultaneous primes

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Theorem (Skorobogatov–Sofos, 2023)

Let $S_{d,N}$ be the set of $f \in P_{d,N}$ such that $X^p - X \nmid f$ in all $\mathbb{F}_p[X]$. Then

$$\lim_{N\to\infty} \frac{\#\left\{(f_1,\ldots,f_k) \in S_{d,N}^k : \begin{array}{c} \exists n \in \mathbb{Z},\ f_1(n),\ldots,f_k(n) \\ \textit{are simultaneously prime} \end{array}\right\}}{\# S_{d,N}^k} = 1.$$

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Theorem (Skorobogatov–Sofos, 2023)

Let K be a cyclic number field with integral basis $e_1,\ldots,e_m$ of $\mathcal{O}_K$. Then

$$\lim_{N\to\infty} \frac{\#\left\{f \in P_{d,N} : \begin{array}{c} \mathrm{Nm}_{\mathbb{Q}}^{K}(e_1X_1 + \cdots + e_mX_m) = f(X) \\ \textit{has a rational point} \end{array}\right\}}{\# P_{d,N}} = 1.$$

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The Hasse principle holds for a variety V over a global field K if it has a point in K whenever it has points in K_v for all places v of K.

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Theorem (Hasse norm theorem)

Let K be a cyclic number field. Then there is a short exact sequence

$$1 \to \mathbb{Q}^\times/\mathrm{Nm}_{\mathbb{Q}}^K(K^\times) \to \bigoplus_{p \leq \infty} \mathbb{Q}_p^\times/\mathrm{Nm}_{\mathbb{Q}}^K((K \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times) \to \mathrm{Gal}(K/\mathbb{Q}) \to 1.$$

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Thus a local norm everywhere except possibly one place is a global norm.

Application of Dirichlet's theorem

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$$(1)^2 + 3(1)^2 \equiv 5(1) + 7 \mod 2^3,$$

$$(3)^2 + 3(1)^2 \equiv 5(1) + 7 \mod 3^3,$$

so it has points in $\mathbb{Q}_2$ and $\mathbb{Q}_3$ by Hensel's lemma.

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$$5x + 7 = 5(2^3 \cdot 3^3 \cdot n + 1) + 7 = 2^2 \cdot 3 \cdot (90n + 1).$$

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By Dirichlet's theorem, there is some n such that $90n + 1$ is prime. For instance, $n = 2$ gives $Y^2 + 3Z^2 = 2^2 \cdot 3 \cdot 181$, which has points in $\mathbb{Q}_2$, $\mathbb{Q}_3$, and $\mathbb{R}$, but also $\mathbb{Q}_p$ for all primes p except 181.

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Let $a_1, \ldots, a_k \in \mathbb{Q}^\times$*, and let* $f_1, \ldots, f_k \in \mathbb{Q}[X]$ *be irreducible. Assume Schinzel's hypothesis H. Then the Hasse principle holds for*

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Let V be the variety over $\mathbb{Q}$ given by $Y^2 + Z^2 = -(X-2)(X-3)$. Then V has points in $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p but no points in $\mathbb{Q}$.

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Let V be the variety over $\mathbb{Q}$ given by $Y^2 + Z^2 = -(X-2)(X-3)$. Then V has points in $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p but no points in $\mathbb{Q}$. The failure of the Hasse principle can be detected by $(3 - X^2, -1) \in \mathrm{Br}(V)[2]$.

The Brauer–Manin obstruction

Let V be a variety over a global field K. There is a commutative diagram

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Example (Iskovskikh, 1971)

Let $A := (3 - X^2, -1) \in \mathrm{Br}(V)$. For any $(x_v)_v \in V(\mathbb{A}_K)$, it can be shown that $\sum_v \mathrm{inv}_v(x_v^* A) = \frac{1}{2}$, so that $V(K) \subseteq V(\mathbb{A}_K)^A = \emptyset$.

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Theorem (Colliot-Thélène–Swinnerton-Dyer, 1994)

Assume Schinzel's hypothesis H. Then Colliot-Thélène's conjecture holds for Severi–Brauer bundles over $\mathbb{P}^1_{\mathbb{Q}}$.