# Teaching a computer algebraic number theory

David Kurniadi Angdinata

London School of Geometry and Number Theory

Tuesday, 25 March 2025

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Today, I will share my thoughts so far:

- why do interactive theorem proving?
- why do interactive theorem proving in Lean?
- why do algebraic number theory in Lean?

# Interactive theorem proving

Wikipedia says that an **interactive theorem prover** is a software tool to assist with the development of formal proofs by human–machine collaboration, which involves some sort of interactive proof editor, or other interface, with which a human can guide the search for proofs, the details of which are stored in, and some steps provided by, a computer.

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In this sense, formalisations in interactive theorem provers are *verified*, in contrast to computer algebra systems like Magma and SageMath.

# Basic example

Here is a theorem in Lean that $\mathrm{im}(f) \leq \ker(f)$ whenever $g \circ f = 1$.

```
variable {A B C : Type} [Group A] [Group B] [Group C]

def ker (f : A →* B) : Subgroup A where
  carrier  := {a : A | f a = 1}
  one_mem' := by simp
  inv_mem' := by simp
  mul_mem' := by aesop

def im (f : A →* B) : Subgroup B where
  carrier  := {b : B | ∃ a : A, b = f a}
  one_mem' := ⟨1, by simp⟩
  inv_mem' := fun ⟨a, _⟩ ↦ ⟨a⁻¹, by aesop⟩
  mul_mem' := fun ⟨a, _⟩ ⟨b, _⟩ ↦ ⟨a * b, by aesop⟩

theorem im_le_ker {f : A →* B} {g : B →* C} (h : ∀ a : A, g (f a) = 1) :
    im f ≤ ker g := by    -- Goal is ∀ b : B, (∃ a : A, b = f a) → (g b = 1)
  intro b hb              -- Goal is g b = 1
                          -- New hypotheses (b : B) (hb : ∃ a : A, (b = f a))
  rcases hb with ⟨a, ha⟩  -- Goal is g b = 1
                          -- New hypotheses (b : B) (a : A) (ha : b = f a)
  rewrite [ha]            -- Goal is g (f a) = 1
  exact h a               -- No goals!
```

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- The proof of the Kepler conjecture by Hales (1998) involved solving 100000 linear programming problems. Hales started the **Flyspeck project** (2003) to formalise it in Coq, and completed it in *11 years*.

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- The polynomial Freiman–Ruzsa conjecture was proven by Gowers, Green, Manners, and Tao (Nov 2023). Tao started a formalisation project in Lean 4 days later, and completed it in *3 weeks*.

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- They present surprising artifacts, such as unnecessary assumptions, simplification of arguments, or even issues in existing literature.
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- They are *fun*! Example: I am addicted.

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Furthermore, it is also a *functional programming language*, written in C++ and *Lean*, with support for *multithreading* and *metaprogramming*.

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Furthermore, it is also a *functional programming language*, written in C++ and *Lean*, with support for *multithreading* and *metaprogramming*.

Finally, it supports *Unicode symbols*, and can be run in *common editors* like Visual Studio Code, Emacs, and Neovim.

# Lean's mathematical community

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- local class field theory by Frutos-Fernández and Nuccio (Sep 2022 –)
- prime number theorem and... led by Kontorovich (Jan 2024 –)
- analytic number theory exponent database led by Tao (Aug 2024 –)
- infinity cosmos led by Riehl (Sep 2024 –)
- Fermat's last theorem led by Buzzard (Oct 2024 –)

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In particular, `mathlib` also knows some non-trivial results in number theory, which is fundamentally *applied* pure mathematics, including:

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It will be learning about class field theory in July!

# Elliptic curves in Lean

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However, `mathlib` does not know what a curve is! This is actually fine, because the Riemann–Roch theorem gives an equivalence of categories

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for any field $F$. Currently `mathlib` thinks that

- an elliptic curve $E$ over a ring $R$ is the data of a tuple $(a_1, a_2, a_3, a_4, a_6) \in R^5$ and a proof that $\Delta(a_i) \in R^\times$, and
- a point on $E$ is either $\mathcal{O}$ or a tuple $(x, y) \in R^2$ such that

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The algebra can be developed independently of the geometry!

How far can we develop the arithmetic purely algebraically?

# The group law in Lean

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$$\begin{array}{rcl} E(F) & \longrightarrow & \mathrm{Cl}(F[E]) \\ \mathcal{O} & \longmapsto & [\langle 1\rangle] \\ (x,y) & \longmapsto & [\langle X-x, Y-y\rangle] \end{array}.$$

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In 2022, Junyan Xu discovered an elementary but novel proof that this map is injective, *due to limitations in* `mathlib`. I formalised his argument in Lean and we wrote a paper that was published in ITP 2023.

# The $n$-torsion subgroup in Lean

The fundamental theorem in the *arithmetic* of elliptic curves is the fact that $E_{\overline{F}}[n]$ is a rank two module over $(\mathbb{Z}/n)[G_F]$ whenever $\mathrm{char}(F) \nmid n$.

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Thus the $\ell$-adic Tate module $T_\ell E_{\overline{F}} := \varprojlim_n E_{\overline{F}}[\ell^n]$ is a two-dimensional $\ell$-adic Galois representation, which is crucial in Tate's isogeny theorem, Serre's open image theorem, Wiles's modularity lifting theorem, etc.

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Exercise 3.7(d) claims that for any point $(x, y) \in E(F)$,

$$[n]((x, y)) = \left(\frac{\phi_n(x, y)}{\psi_n(x, y)^2}, \frac{\omega_n(x, y)}{\psi_n(x, y)^3}\right).$$

The key idea is that $\psi_n(x, y) = 0$ occurs precisely when $[n]((x, y)) = \mathcal{O}$.

# The multiplication-by-$n$ formula in Lean

**Conjecture**

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- Definition of $\omega_n$ is *incorrect*! It should instead be

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- Exercise 3.7(d) needs four special cases of this stronger result. $\square$

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$$\begin{array}{c}
\text{Def of } \psi_n \text{ (3.7(a))} \\
\downarrow \\
\text{Def of } \phi_n \text{ (3.7(a))} \\
\downarrow \\
\deg(\phi_n^2) = n^2 \text{ (3.7(b))}
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Projective coordinates

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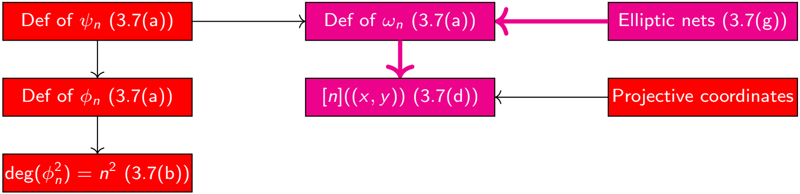

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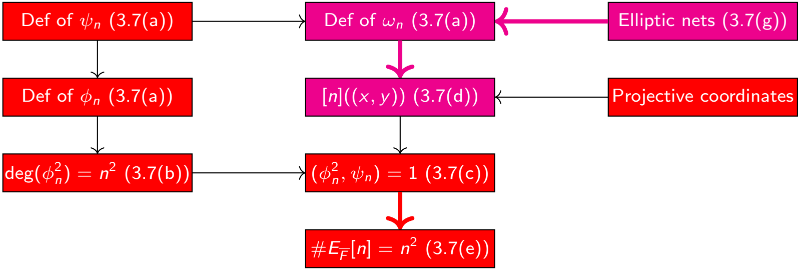

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\begin{tikzcd}
\text{Def of } \psi_n \text{ (3.7(a))} \arrow[r] \arrow[d] & \text{Def of } \omega_n \text{ (3.7(a))} \arrow[d] & \text{Elliptic nets (3.7(g))} \arrow[l] \\
\text{Def of } \phi_n \text{ (3.7(a))} \arrow[d] & {[n]((x,y))} \text{ (3.7(d))} \arrow[d] & \text{Projective coordinates} \arrow[l] \\
\deg(\phi_n^2) = n^2 \text{ (3.7(b))} \arrow[r] & (\phi_n^2, \psi_n) = 1 \text{ (3.7(c))} \arrow[d] & \\
& \# E_{\overline{F}}[n] = n^2 \text{ (3.7(e))} \arrow[d] & \\
& E_{\overline{F}}[n] \cong (\mathbb{Z}/n)^2 \arrow[d] & \text{Finite abelian groups} \arrow[l] \\
& T_\ell E_{\overline{F}} \cong \mathbb{Z}_\ell^2 & \text{Inverse limits} \arrow[l]
\end{tikzcd}
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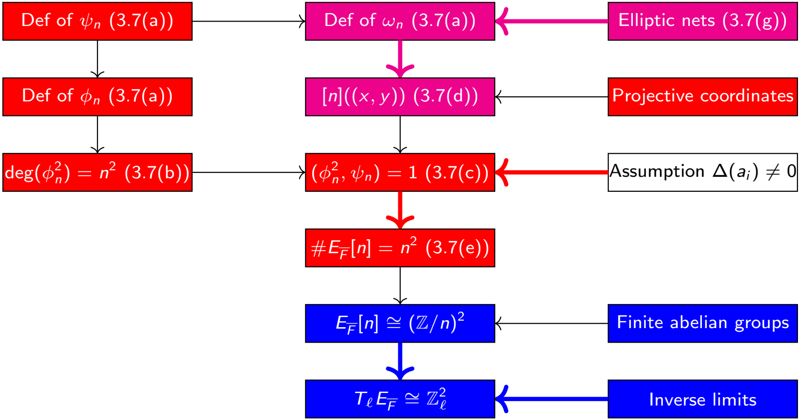


Note that the assumption $\Delta(a_i) \neq 0$ is unnecessary until Exercise 3.7(c).

# Future projects

Can we formalise the *full statement* of the Birch and Swinnerton-Dyer conjecture for an elliptic curve $E$ over a number field $K$ in Lean?

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L(E, s) &
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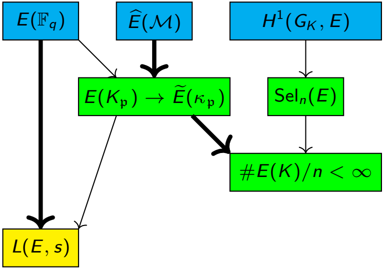

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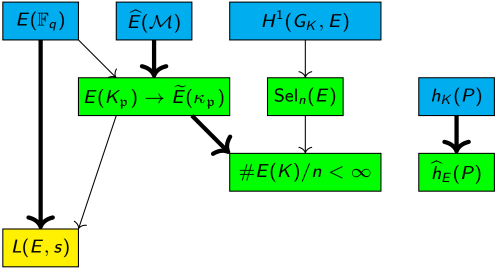

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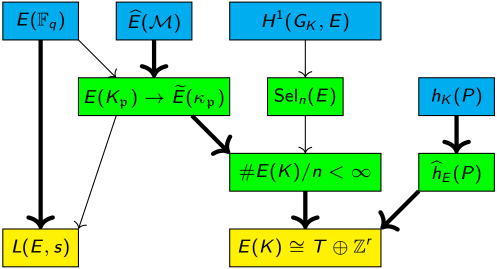

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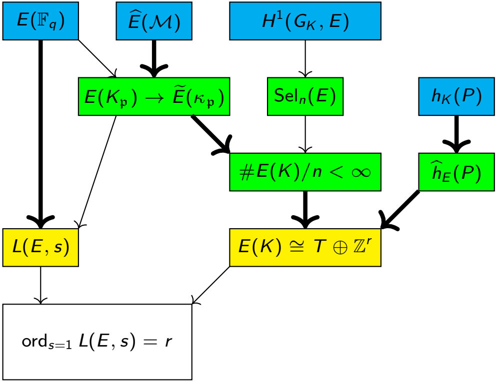

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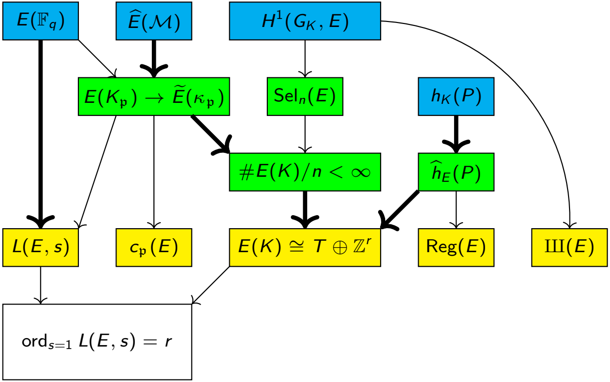

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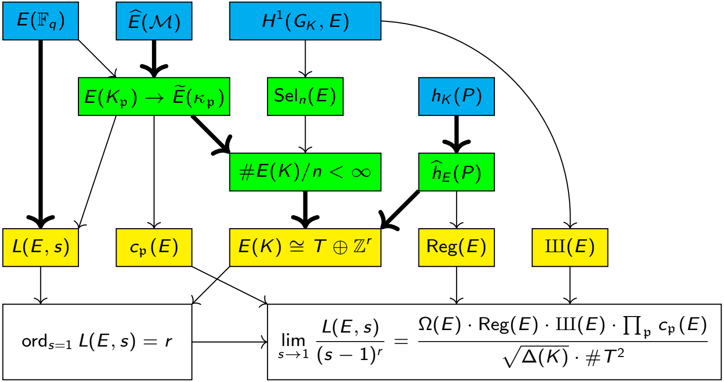

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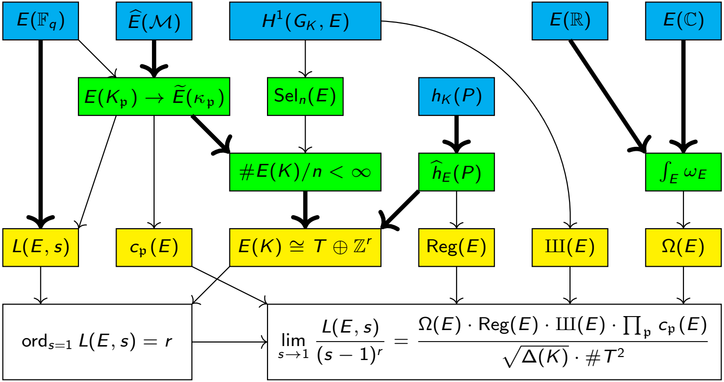

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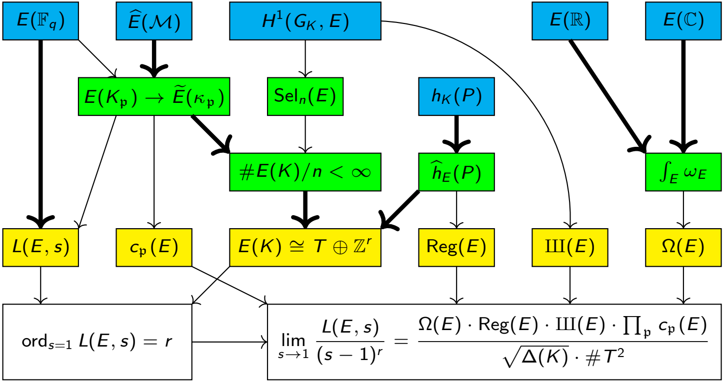


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