Twisted elliptic L-values

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In particular, $E(K) \cong \mathrm{tor}_{E/K} \times \mathbb{Z}^{\mathrm{rk}_{E/K}}$, where

- $\mathrm{tor}_{E/K}$ is the *torsion subgroup*, and
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While $\mathrm{tor}_{E/K}$ is classified, $\mathrm{rk}_{E/K}$ remains mysterious.

Conjecture (Birch–Swinnerton-Dyer, weak form)

The order of vanishing of $L_{E/K}(s)$ at $s = 1$ is equal to $\mathrm{rk}_{E/K}$.

This is called the *analytic rank*.

L-functions

For any G_K-representation ρ, its **local Euler factor** is given by

$$L_{\mathfrak{p}}(\rho, T) := \det(1 - T \cdot \phi_{\mathfrak{p}} \mid \rho^{I_{\mathfrak{p}}}),$$

where $\phi_{\mathfrak{p}} \in G_K$ is a Frobenius and $I_{\mathfrak{p}} \leq G_K$ is the inertia subgroup.

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The **(Hasse-Weil) L-function of** E/K is given by

$$L_{E/K}(s) := \prod_{\mathfrak{p}} \frac{1}{L_{\mathfrak{p}}(\rho_{E,\ell}, \mathrm{Nm}(\mathfrak{p})^{-s})},$$

where $\rho_{E,\ell}$ is the rational ℓ-adic Tate module as a G_K-representation.

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Example ($K = \mathbb{Q}$)

Let $a_p := 1 + p - \#E(\mathbb{F}_p)$. Then

$$L_p(\rho_{E,\ell}, p^{-s}) = \begin{cases} 1 - a_p p^{-s} + p^{1-2s} & p \nmid \Delta(E) \\ 1 - a_p p^{-s} & p \mid \Delta(E) \end{cases}.$$

The BSD conjecture

Conjecture (Birch–Swinnerton-Dyer, strong form)

The leading term of $L_{E/K}(s)$ at $s = 1$ satisfies

$$\lim_{s\to 1} \frac{L_{E/K}(s)}{(s-1)^{\mathrm{rk}_{E/K}}} \cdot \frac{\sqrt{|\Delta_K|}}{\Omega_{E/K}} = \frac{C_{E/K} \cdot R_{E/K} \cdot \#\text{Ш}_{E/K}}{\#\mathrm{tor}^2_{E/K}}.$$

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Here,

- $\Omega_{E/K}$ is the *global period*,
- $C_{E/K}$ is the *Tamagawa product*,
- $R_{E/K}$ is the *regulator*, where $R_{E/K} = 1$ if $\mathrm{rk}_{E/K} = 0$, and
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If $\mathrm{rk}_{E/K} = 0$, the LHS is called the **algebraic L-value**, given by

$$\mathcal{L}_{E/K} := L_{E/K}(1) \cdot \frac{\sqrt{\Delta_K}}{\Omega_{E/K}}.$$

Twisted L-functions

Let $K = \mathbb{Q}(\zeta_m)$, and let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character.

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Note that

$$L_E(s) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^s} \quad \xrightarrow{\chi} \quad L_{E,\chi}(s) = \sum_{n\in\mathbb{N}} \frac{a_n \chi(n)}{n^s}.$$

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By representation theory, there is a factorisation

$$L_{E/K}(s) = \prod_\chi L_{E,\chi}(s),$$

where $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ runs over primitive Dirichlet characters.

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Conjecture (Deligne–Gross)

The order of vanishing of $L_{E,\chi}(s)$ *at* $s = 1$ *is equal to* $\langle \chi, E(K)_{\mathbb{C}} \rangle$.

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Example (Dokchitser–Evans–Wiersema)

Let E_1 and E_2 be elliptic curves given by Cremona labels 307a1 and 307c1, and let $\chi : (\mathbb{Z}/11\mathbb{Z})^\times \to \mathbb{C}^\times$ be the primitive Dirichlet character of order 5 and conductor 11 given by $\chi(2) = \zeta_5$.

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$$C_{E_i/K} = R_{E_i/K} = \text{Ш}_{E_i/K} = \mathrm{tor}_{E_i/K} = 1,$$

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for $K \subseteq \mathbb{Q}(\zeta_{11})^+$, but

$$\mathcal{L}_{E_1,\chi} = 1, \qquad \mathcal{L}_{E_2,\chi} = \zeta_5(1 + \zeta_5^4)^2.$$

Algebraic twisted L-values

If $\mathrm{rk}_E = 0$, the **algebraic twisted L-value** is given by

$$\mathcal{L}_{E,\chi} := L_{E,\chi}(1) \cdot \frac{\tau(\overline{\chi})}{\Omega_E},$$

where $\tau(\overline{\chi})$ is the **Gauss sum**

$$\tau(\overline{\chi}) := \sum_{n \in (\mathbb{Z}/m\mathbb{Z})^\times} \overline{\chi}(n) \zeta_m^n.$$

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In general $\mathcal{L}_{E,\chi} \in \overline{\mathbb{Q}}$, but some integrality statements are known.

Theorem (Wiersema–Wuthrich)

If E is semistable optimal of conductor N_E, and if χ is primitive of order k and conductor coprime to N_E, then $\mathcal{L}_{E,\chi} \in \mathbb{Z}[\zeta_k]$.

There are stronger statements under the *Manin constant conjecture*.

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Lemma (Kisilevsky–Nam)

Let ω_E be the "root number" of E. Then $\lambda_\chi \cdot \mathcal{L}_{E,\chi} \in \mathbb{Z}[\zeta_k]^+$, where

$$\lambda_\chi := \begin{cases} 1 & \omega_E \cdot \chi(-N_E) = 1 \\ \chi(m) - \overline{\chi(m)} & \omega_E \cdot \chi(-N_E) = -1 \\ 1 + \omega_E \cdot \overline{\chi(-N_E)} & \omega_E \cdot \chi(-N_E) \neq \pm 1 \end{cases}, \qquad m \in \mathbb{Z}.$$

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Example ($k = 3$)

$$\mathbb{Z} \ni \mathcal{L}^+_{E,\chi} = \begin{cases} \Re(\mathcal{L}_{E,\chi}) & \omega_E \cdot \chi(N_E) = 1 \\ 2\Re(\mathcal{L}_{E,\chi}) & \omega_E \cdot \chi(N_E) \neq 1 \end{cases}.$$

Some observations

Let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ run over primes $p \equiv 1 \mod 3$.

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Let E be the elliptic curve given by the Cremona label 11a1.

p	7	13	19	31	37	43	61	67	73	79	97
$\mathcal{L}^+_{E,\chi}$	5	-10	-10	5	20	5	-10	15	5	15	-30

p	103	109	127	139	151	157	163	181	193	199
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If E is the elliptic curve given by the Cremona label 11a1,

$$\overline{\mathcal{L}}_{E,\chi}^{+} \equiv_3 \begin{cases} 0 & \#E(\mathbb{F}_p) \equiv 0 \mod 3 \\ 2 & \#E(\mathbb{F}_p) \equiv 1 \mod 3 \text{ and } \chi(N_E) = 1 \\ 1 & \text{otherwise} \end{cases}.$$

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Theorem (A.)

I can partially explain the DEW and KN phenomena.

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For any cusp form $f \in S_k(\Gamma)$, its L-function is a Mellin transform

$$L_f(s) := \frac{(-2\pi i)^s}{\Gamma(s)} \int_0^{i\infty} z^{s-1} f(z) \mathrm{d}z.$$

The modularity theorem

Let E be a semistable optimal elliptic curve over $\mathbb{Q}$ of conductor N_E.

Theorem (Taylor–Wiles)

There is an eigenform $f_E \in S_2(\Gamma_0(N_E))$ with (Hecke) L-function $L_{f_E}(s) = L_E(s)$, such that the Hecke operator T_p has eigenvalue a_p.

For any cusp form $f \in S_k(\Gamma)$, its L-function is a Mellin transform

$$L_f(s) := \frac{(-2\pi i)^s}{\Gamma(s)} \int_0^\infty z^{s-1} f(z) \mathrm{d}z.$$

Set $s = 1$:

$$L_f(1) = -2\pi i \int_0^\infty f(z) \mathrm{d}z =: -\langle \{0, \infty\}, f \rangle.$$

This is a *period* of the *modular symbol* $\{0, \infty\}$.

Classical modular symbols

Let $\mathcal{H}$ be the extended upper half plane, and let $\phi : \mathcal{H} \twoheadrightarrow \mathcal{H}/\Gamma =: X_\Gamma$.

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- If $\Gamma \cdot x = \Gamma \cdot y$, then $\phi(x \rightsquigarrow y) \in H_1(X_\Gamma, \mathbb{Z})$, and conversely any $\gamma \in H_1(X_\Gamma, \mathbb{Z})$ arises from $x, y \in \mathcal{H}$ in the same Γ-orbit. Define

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$$\{x, y\} := \phi(x \rightsquigarrow y).$$
- The map $H_1(X_\Gamma, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma), \mathbb{C})$ given by $\gamma \mapsto \langle \gamma, \cdot \rangle$ extends to $\psi : H_1(X_\Gamma, \mathbb{R}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma), \mathbb{C})$. Define
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Note that

$$\{x, x\} = 0, \quad \{x, y\} = -\{y, x\}, \quad \{x, y\} + \{y, z\} = \{x, z\},$$

but also

$$\langle \{x, y\}, M \cdot f \rangle = \langle \{M \cdot x, M \cdot y\}, f \rangle, \qquad M \in \Gamma.$$

L-values as periods

Let $p \nmid N_E$. The Hecke operator T_p acts on $H_1(X_\Gamma, \mathbb{Q})$ by

$$T_p \cdot \{x, y\} = \{px, py\} + \sum_{n=0}^{p-1} \{\tfrac{x+n}{p}, \tfrac{y+n}{p}\}.$$

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Lemma (Manin)

$$-\#E(\mathbb{F}_p) \cdot \mathcal{L}_E = \frac{1}{\Omega_E} \sum_{n=1}^{p-1} \langle \{0, \tfrac{n}{p}\}, f_E \rangle.$$

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Proof.

Set $\{x, y\} = \{0, \infty\}$ in the Hecke action and apply the pairing $\langle \cdot, f_E \rangle$:

$$(1 + p - a_p) \cdot \langle \{0, \infty\}, f_E \rangle = \sum_{n=1}^{p-1} \langle \{0, \tfrac{n}{p}\}, f_E \rangle.$$

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Multiply by $\frac{1}{\Omega_E}$. □

Twisted L-values as periods

Let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character.

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$$\mathcal{L}_{E,\chi} = \frac{1}{\Omega_E} \sum_{n=1}^{p-1} \overline{\chi}(n) \langle \{0, \tfrac{n}{p}\}, f_E \rangle.$$

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Multiply by $\frac{\tau(\overline{\chi})}{\Omega_E}$. □

A congruence for L-values

Let E be a semistable optimal elliptic curve over $\mathbb{Q}$ of conductor N_E, let $p \nmid N_E$, and let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character.

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Corollary (A)

If χ has prime order k, then

$$-\#E(\mathbb{F}_p) \cdot \mathcal{L}_E \equiv \mathcal{L}_{E,\chi} \mod (1 - \zeta_k).$$

Proof.

By integrality, the lemmata, and $\overline{\chi} \equiv 1 \mod (1 - \zeta_k)$. □

A congruence for cubic twists

Let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ be a cubic primitive Dirichlet character.

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Corollary (B)

$$\mathcal{L}^+_{E,\chi} \equiv_3 \#E(\mathbb{F}_p) \cdot \mathcal{L}_E \cdot \begin{cases} 2 & \omega_E \cdot \chi(N_E) = 1 \\ 1 & \omega_E \cdot \chi(N_E) \neq 1 \end{cases}.$$

Proof.

By cases of corollary (A). □

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By cases of corollary (A). □

Corollary (C)

$$\overline{\mathcal{L}}_{E,\chi}^+ \equiv_3 \#E(\mathbb{F}_p) \cdot \mathcal{L}_E \cdot \gcd_{\chi'}\{\mathcal{L}_{E,\chi'}^+\} \cdot \begin{cases} 2 & \omega_E \cdot \chi(N_E) = 1 \\ 1 & \omega_E \cdot \chi(N_E) \neq 1 \end{cases}.$$

Proof.

By cases of corollary (B). □

DEW phenomena

Recall that if E_1 and E_2 are elliptic curves given by Cremona labels 307a1 and 307c1, and $\chi : (\mathbb{Z}/11\mathbb{Z})^\times \to \mathbb{C}^\times$ is the primitive Dirichlet character of order 5 and conductor 11 given by $\chi(2) = \zeta_5$, then

$$C_{E_i/K} = R_{E_i/K} = \text{Ш}_{E_i/K} = \mathrm{tor}_{E_i/K} = 1,$$

for $K \subseteq \mathbb{Q}(\zeta_{11})^+$, but

$$\mathcal{L}_{E_1,\chi} = 1, \qquad \mathcal{L}_{E_2,\chi} = \zeta_5(1+\zeta_5^4)^2.$$

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Note that $\mathcal{L}_{E_i} = 1$, and

$$\#E_1(\mathbb{F}_{11}) = 9, \qquad \#E_2(\mathbb{F}_{11}) = 16,$$

so corollary (A) says

$$\mathcal{L}_{E_1,\chi} \not\equiv \mathcal{L}_{E_2,\chi} \mod (1 - \zeta_5).$$

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In fact, corollary (A) partially explains all examples in DEW where χ is quintic, and fully explains all examples in DEW where χ is cubic.

Insufficiency of congruence

Unfortunately, there are elliptic curves E_1 and E_2 over $\mathbb{Q}$, where $\mathcal{L}_{E_1,\chi} \equiv \mathcal{L}_{E_2,\chi} \mod (1-\zeta_5)$, but $\mathcal{L}_{E_1,\chi} \neq \mathcal{L}_{E_2,\chi}$.

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Example (A.)

Let E_1 and E_2 be elliptic curves given by Cremona labels 130b3 and 312c3, and let $\chi : (\mathbb{Z}/11\mathbb{Z})^\times \to \mathbb{C}^\times$ be the primitive Dirichlet character of order 5 and conductor 11 given by $\chi(2) = \zeta_5$.

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Heuristically, the norm of $\overline{\mathcal{L}}^+_{E,\chi}$ is the χ-component of $Ш_E$.

KN phenomena

Recall that if E is the elliptic curve given by the Cremona label 11a1,

$$\overline{\mathcal{L}}_{E,\chi}^{+} \equiv_3 \begin{cases} 0 & \#E(\mathbb{F}_p) \equiv 0 \mod 3 \\ 2 & \#E(\mathbb{F}_p) \equiv 1 \mod 3 \text{ and } \chi(N_E) = 1 \\ 1 & \text{otherwise} \end{cases}.$$

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This fully explains the three cases, except for

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In fact, corollary (C) fully explains $\overline{\mathcal{L}}_{E,\chi}^{+}$ modulo 3 for any E where

- E does not have rational 3-isogenies, and
- the 3-division field $\mathbb{Q}(x(E[3]))$ of E contains $\sqrt[3]{N_E}$.

The missing piece

Theorem (A.–Dokchitser)

Assume that

- *E does not have rational* 3*-isogenies, and*
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This crucially uses the classification of 3-adic images of Galois for elliptic curves over $\mathbb{Q}$ by Rouse–Sutherland–Zureick-Brown.

Future work

Here are some potential extensions, listed in increasing difficulty:

- replace $\mathbb{Q}$ with a global field
- replace χ with an Artin representation
- replace E with the Jacobian of a higher genus curve
- remove the $\mathrm{rk}_E = 0$ assumption

Thank you!