Twisted elliptic L-values over global fields

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Algebraic Number Theory

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where $L_{\mathfrak{p}}(\rho, T) := \det(1 - T \cdot \phi_{\mathfrak{p}} \mid \rho^{I_{\mathfrak{p}}})$.

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If L is a finite Galois extension of K with Galois group G, then

$$L(A/L, s) = L(A, \mathrm{Ind}_{\{1\}}^{G} \mathbb{1}, s) = \prod_{\chi} L(A, \chi, s),$$

where χ runs over all the irreducible characters $\mathrm{Irr}(G)$ of G.

The Birch–Swinnerton-Dyer conjecture

Conjecture (Birch–Swinnerton-Dyer)

The order of vanishing of $L(A, s)$ *at* $s = 1$ *is equal to* $\mathrm{rk}(A)$*.*
Furthermore, the leading term of $L(A, s)$ *at* $s = 1$ *is equal to*

$$L^*(A, 1) = \frac{\Omega(A) \cdot \mathrm{Reg}(A) \cdot \mathrm{Tam}(A) \cdot \#\text{Ш}(A)}{\sqrt{|\Delta_K|} \cdot \#\mathrm{tor}(A) \cdot \#\mathrm{tor}(\widehat{A})}.$$

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This is known in some cases when A is an elliptic curve.

- If $K = \mathbb{Q}$ and the order of vanishing is at most 1, then the rank conjecture is proven by Gross–Zagier 1986 and Kolyvagin 1988, and much of the ℓ-part of the leading term conjecture is proven by Keller–Yin 2024 and Burungale–Castella–Skinner 2024.
- If $K = \mathbb{F}_p(C)$, then Kato–Trihan 2003 proved that the rank conjecture is equivalent to the finiteness of $\text{Ш}(A)[\ell^\infty]$ for some prime $\ell \neq p$ and implies the leading term conjecture.

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This is known in some cases when A is an elliptic curve over $\mathbb{Q}$.

- If A has no potential complex multiplication, then Kato 2004 proved this for one-dimensional Artin representations.
- If the order of vanishing is 0, then Bertolini–Darmon–Rotger 2015 proved this for odd irreducible two-dimensional Artin representations.
- If the order of vanishing is 0, then Darmon–Rotger 2017 proved this for certain self-dual Artin representations of dimension at most 4.

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Theorem (Bisatt–Dokchitser 2018)

Assume the Deligne–Gross conjecture. If $\chi \in \mathrm{Irr}(C_q \rtimes C_{p^n})$ *with* $q \not\equiv 1 \mod p^n$, *then* p *divides the order of vanishing of* $L(A, \chi, s)$ *at* $s = 1$.

A twisted leading term conjecture

There seems to be a barrier to a leading term conjecture for $L(A, \chi, s)$.

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Example

Let A_1 and A_2 be elliptic curves over $\mathbb{Q}$ given by Cremona labels 1356d1 and 1356f1, and let χ be the primitive Dirichlet character of conductor 7 and order 3 given by $\chi(3) = \zeta_3^2$.

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$$\mathrm{Reg}(A_i/K) = \mathrm{Tam}(A_i/K) = \text{Ш}(A_i/K) = \mathrm{tor}(A_i/K) = 1,$$

for $K = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_7)^+$, but $\mathcal{L}(A_1, \chi) = \zeta_3^2$ and $\mathcal{L}(A_2, \chi) = -\zeta_3^2$.

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Theorem (Dokchitser–Evans–Wiersema 2021)

Assume there is a conjecture $\mathcal{L}(A,\chi)=\mathrm{BSD}(A,\chi)$ for a semistable elliptic curve A over $\mathbb{Q}$. If $\chi\in\mathrm{Irr}(D_{pq})$ with $p\equiv q\equiv 3 \mod 4$, then $\langle\chi, A(L)_{\mathbb{C}}\rangle>0$ if the order of vanishing of $L(A,\chi,s)$ at $s=1$ is odd.

Algebraic L-values

Define the **algebraic L-value** of A by

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If A is an elliptic curve over $\mathbb{Q}$, then modularity gives

$$-(1+p-a_p(A))\cdot L(f_A,1) = \sum_{n=1}^{p-1}\int_0^{\frac{n}{p}} f_A(q)\mathrm{d}q,$$

which is a rational multiple of $\Omega(A)$.

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In general, the algebraicity of $\mathcal{L}(A)$ is Deligne's period conjecture.

Deligne's period conjecture

A motive M over a global field K is a collection of K-vector space realisations $H_B(M)$, $H_{dR}(M)$, $H_\lambda(M)$, and $H_p(M)$, equipped with comparison isomorphisms between their complexifications.

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Conjecture (Deligne)

Let M be a critical motive over a number field K such that $L(M,0) \neq 0$. Then there is some $x \in K^\times$ such that

$$L(M,0) = x^\sigma \cdot c^+(M), \qquad \sigma \in \mathrm{Gal}(K/\mathbb{Q}).$$

Here, $c^+(M)$ is the determinant of the period map

$$H_B(M)^+ \otimes \mathbb{C} \hookrightarrow H_B(M) \otimes \mathbb{C} \xrightarrow{\sim} H_{dR}(M) \otimes \mathbb{C} \twoheadrightarrow H_{dR}(M)^+ \otimes \mathbb{C}.$$

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If $M = h^1(A)(1)$, then this says that there is some $x \in \mathbb{Q}^\times$ such that

$$L(A,1) = x \cdot \Omega(A).$$

Algebraic twisted L-values

Define the **algebraic twisted L-value** of (A, χ) by

$$\mathcal{L}(A, \chi) := \frac{L^*(A, \chi, 1)}{\Omega(A, \chi) \cdot \mathrm{Reg}(A, \chi)}.$$

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Define the twisted period of (A, χ) by

$$\Omega(A,\chi) := \frac{\Omega_+(A)^{\dim^+(\chi)}\cdot \Omega_-(A)^{\dim^-(\chi)}\cdot w_\chi^{\dim(A)}}{\sqrt{N_\chi}^{\dim(A)}}.$$

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Define the twisted regulator of (A, χ) by

$$\mathrm{Reg}(A, \chi) := \det(\langle e_i(\chi), e_j(\widehat{\chi})\rangle)_{i,j},$$

where $\{e_i(\chi)\}_i$ is a basis of

$$A(L)[\chi] := \mathrm{Hom}_{\mathbb{Z}[\chi]}(\rho_\chi, A(L) \otimes_{\mathbb{Z}} \mathbb{Z}[\chi])^{\mathrm{Gal}(L/K)}.$$

Algebraicity of twisted L-values

If $L(A, \chi, 1) \neq 0$, then $\mathrm{Reg}(A, \chi) = 1$. Then Deligne's period conjecture for $M = h^1(A)(1) \otimes \chi$ says that there is some $x \in \mathbb{Q}(\chi)^\times$ such that

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Theorem (Bouganis–Dokchitser 2007, Wiersema–Wuthrich 2021)

Let L be a finite abelian extension of $\mathbb{Q}$ with Galois group G, and let A be an elliptic curve over $\mathbb{Q}$ such that $L(A,\chi,1) \neq 0$. Then for any non-trivial $\chi \in \mathrm{Irr}(G)$ such that $(N_\chi, N_A) = 1$,

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Furthermore, $\mathcal{L}(A,\chi) \in \mathbb{Z}[\chi]$ assuming that $c_1(A) = 1$.

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Castillo–Evans–Wiersema 2023 gave numerical evidence for $A = \mathrm{Jac}(C)$.

Ideals of twisted L-values

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Let L be a finite Galois extension of $\mathbb{Q}$ with Galois group G, and let A be an abelian variety over $\mathbb{Q}$ such that $(\Delta_L, N_A) = 1$. Assume that the refined Birch–Swinnerton-Dyer conjecture holds for $(A, L, \mathbb{Q})$. Let $\chi \in \mathrm{Irr}(G)$, and let λ be a prime of $\mathbb{Q}(\chi)$ not dividing

$$2, \quad |G|, \quad \Delta_L, \quad N_A, \quad \mathrm{Tam}(A), \quad \#\mathrm{tor}(A/L), \quad \#\mathrm{tor}(\widehat{A}/L).$$

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Then there is an equality of fractional $\mathbb{Z}[\chi]_\lambda$-ideals

$$\mathcal{L}(A,\chi)\cdot\mathbb{Z}[\chi]_\lambda = \frac{\mathrm{char}_\lambda(\text{Ш}(A/L,\chi))}{\prod_{v|\Delta_L} L_v(A,\chi,1)}.$$

Here, $\text{Ш}(A/L,\chi) := \mathrm{Hom}_{\mathbb{Z}[\chi]}(\rho_\chi, \text{Ш}(A/L)\otimes_{\mathbb{Z}}\mathbb{Z}[\chi])^{\mathrm{Gal}(L/K)}$.

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The norm of $\mathcal{L}(A,\chi)$ has a conjectural expression when $L(A,\chi,1) \neq 0$.

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Let L be a finite abelian extension of $\mathbb{Q}$ with Galois group G, and let A be an elliptic curve over $\mathbb{Q}$ such that $c_1(A) = 1$. Assume that the Birch–Swinnerton-Dyer conjecture holds for (A, L) and $(A, \mathbb{Q})$. Let $\chi \in \mathrm{Irr}(G)$ have odd prime conductor $p \nmid N_A$ and odd prime order $q \nmid \#A(\mathbb{F}_p) \cdot \mathcal{L}(A)$ such that $L(A,\chi,1) \neq 0$.

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$$\mathrm{Nm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_q)^+}(\mathcal{L}(A,\chi)\cdot\zeta)=B(K),$$

where $\zeta:=\chi(N_A)^{(q-1)/2}$ and K is the subfield of $\mathbb{Q}(\zeta_p)$ cut out by χ.

Here,

$$B(K):=\frac{\#\mathrm{tor}(A)}{\#\mathrm{tor}(A/K)}\sqrt{\frac{\mathrm{Tam}(A/K)\cdot\#\text{Ш}(A/K)}{\mathrm{Tam}(A)\cdot\#\text{Ш}(A)}}.$$

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$$\mathcal{L}(A,\chi)\cdot\zeta = \begin{cases} B(K) & \text{if } \#A(\mathbb{F}_p)\cdot\mathcal{L}(A)\cdot B(K)^{-1} \equiv 2 \mod 3 \\ -B(K) & \text{if } \#A(\mathbb{F}_p)\cdot\mathcal{L}(A)\cdot B(K)^{-1} \equiv 1 \mod 3 \end{cases},$$

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where $\zeta := \chi(N_A)^{(q-1)/2}$.

This follows from $-\#A(\mathbb{F}_p)\cdot\mathcal{L}(A) \equiv \mathcal{L}(A,\chi) \mod (1-\zeta_q)$, which arises from a congruence in Manin's formalism for classical modular symbols.

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On the other hand $\#A_1(\mathbb{F}_7) = 11$ and $\#A_2(\mathbb{F}_7) = 7$, so A. 2023 says that

$$\mathcal{L}(A_1, \chi) \equiv -\#A_1(\mathbb{F}_7) \equiv 1 \equiv \zeta_3^2 \mod (1 - \zeta_3),$$

$$\mathcal{L}(A_2, \chi) \equiv -\#A_2(\mathbb{F}_7) \equiv -1 \equiv -\zeta_3^2 \mod (1 - \zeta_3).$$

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Now let A be an abelian variety over a global function field $K = \mathbb{F}_p(C)$.

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$$L(A, \chi, s) = \prod_{i=0}^{2} \det(1 - p^{-s} \cdot \phi_p \mid H^i_{\text{ét},c}(\overline{C}, \mathcal{F}))^{(-1)^{i+1}},$$

where $\mathcal{F}$ is the constructible sheaf on C given by the pushforward of the lisse sheaf $V_\ell(A) \otimes \rho_\chi$ defined over any unramified open subset of C.

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where $\mathcal{F}$ is the constructible sheaf on C given by the pushforward of the lisse sheaf $V_\ell(A) \otimes \rho_\chi$ defined over any unramified open subset of C.

Since $L(A, \chi, s)$ is already algebraic, define $\mathcal{L}(A, \chi) := L^*(A, \chi, 1)$.

Algebraic twisted L-values

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Theorem (A. 2024)

Let L be a finite Galois extension of $K = \mathbb{F}_p(C)$ with Galois group G, and let A be an abelian variety over K. Then for any $\chi \in \mathrm{Irr}(G)$,

$$\mathcal{L}(A, \chi^\sigma) = \mathcal{L}(A, \chi)^\sigma, \qquad \sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}).$$

Ideals of twisted L-values

The ideal generated by $\mathcal{L}(A, \chi)$ has a conjectural twisted BSD formula.

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Let L be a finite Galois extension of $K = \mathbb{F}_p(C)$ with Galois group G, and let A be an abelian variety over K. Assume that $\text{Ш}(A/L)$ is finite. Let $\chi \in \mathrm{Irr}(G)$, and let λ be a prime of $\mathbb{Q}(\chi)$ not dividing

$$p, \qquad |G|, \qquad \mathrm{Tam}(A), \qquad \#\mathrm{tor}(A/L), \qquad \#\mathrm{tor}(\widehat{A}/L).$$

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Then there is an equality of fractional $\mathbb{Z}[\chi]_\lambda$-ideals

$$\mathcal{L}(A,\chi)\cdot\mathbb{Z}[\chi]_\lambda = \frac{\mathrm{Reg}_\lambda(A,\chi)\cdot \mathrm{char}_\lambda(Ш_\lambda(A/L,\chi))}{\prod_{v|\Delta_L} L_v(A,\chi,1)}.$$

This involves the $\mathbb{Z}[\chi]_\lambda$-modules $\mathrm{Reg}_\lambda(A,\chi)$ and $Ш_\lambda(A/L,\chi)$, which are necessary to generalise the statement to primes λ of $\mathbb{Q}(\chi)$ dividing p.

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On the other hand, $L(A,\chi,s)$ is already a rational function in p^{-s} with coefficients in $\mathbb{Q}(\chi)$, which can be determined by investigating the action of ϕ_p on $H^i_{\text{ét},c}(\overline{C},\mathcal{F})$.

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Can we understand the action of ϕ_p from the geometry of (A, χ)?