Twisted elliptic L-values

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A tale of two ranks

Let E be an elliptic curve over $\mathbb{Q}$, and let K be a number field.

Theorem (Mordell–Weil)

The set of K-points $E(K)$ is a finitely generated abelian group.

In particular, $E(K) \cong \mathrm{tor}_{E/K} \times \mathbb{Z}^{\mathrm{rk}_{E/K}}$, where

- $\mathrm{tor}_{E/K}$ is the *torsion subgroup*, and
- $\mathrm{rk}_{E/K}$ is the *(algebraic) rank*.

While $\mathrm{tor}_{E/K}$ is classified, $\mathrm{rk}_{E/K}$ remains mysterious.

Conjecture (Birch–Swinnerton-Dyer, weak form)

The order of vanishing of $L_{E/K}(s)$ at $s = 1$ is equal to $\mathrm{rk}_{E/K}$.

This is called the *analytic rank*.

L-functions

For any G_K-representation ρ, its **local Euler factor** is given by

$$L_{\mathfrak{p}}(\rho, T) := \det(1 - T \cdot \phi_{\mathfrak{p}} \mid \rho^{I_{\mathfrak{p}}}),$$

where $\phi_{\mathfrak{p}} \in G_K$ is a Frobenius and $I_{\mathfrak{p}} \leq G_K$ is the inertia subgroup. The **(Hasse–Weil) L-function of** E/K is given by

$$L_{E/K}(s) := \prod_{\mathfrak{p}} \frac{1}{L_{\mathfrak{p}}(\rho_{E,\ell}, \mathrm{Nm}(\mathfrak{p})^{-s})},$$

where $\rho_{E,\ell}$ is the rational ℓ-adic Tate module as a G_K-representation.

Example ($K = \mathbb{Q}$)

Let $a_p := 1 + p - \#E(\mathbb{F}_p)$. Then

$$L_p(\rho_{E,\ell}, p^{-s}) = \begin{cases} 1 - a_p p^{-s} + p^{1-2s} & \text{if } p \nmid \Delta(E), \\ 1 - a_p p^{-s} & \text{if } p \mid \Delta(E). \end{cases}$$

The BSD conjecture

Conjecture (Birch–Swinnerton-Dyer, strong form)

The leading term of $L_{E/K}(s)$ *at* $s = 1$ *satisfies*

$$\lim_{s \to 1} \frac{L_{E/K}(s)}{(s-1)^{\mathrm{rk}_{E/K}}} \cdot \frac{\sqrt{|\Delta_K|}}{\Omega_{E/K}} = \frac{C_{E/K} \cdot R_{E/K} \cdot \# Ш_{E/K}}{\# \operatorname{tor}_{E/K}^2}.$$

Here,

- $\Omega_{E/K}$ is the *global period*,
- $C_{E/K}$ is the *Tamagawa product*,
- $R_{E/K}$ is the *regulator*, where $R_{E/K} = 1$ if $\mathrm{rk}_{E/K} = 0$, and
- $Ш_{E/K}$ is the *Tate–Shafarevich group*, conjecturally finite.

If $\mathrm{rk}_{E/K} = 0$, the LHS is called the **algebraic L-value**, given by

$$\mathscr{L}_{E/K} := L_{E/K}(1) \cdot \frac{\sqrt{\Delta_K}}{\Omega_{E/K}}.$$

Twisted L-functions

Let $K = \mathbb{Q}(\zeta_m)$, and let $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character.

The **(Hasse–Weil) L-function of E twisted by χ** is given by

$$L_{E,\chi}(s) := \prod_p \frac{1}{L_p(\rho_{E,\ell} \otimes \chi, p^{-s})}.$$

Note that

$$L_E(s) = \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} \quad \overset{\chi}{\rightsquigarrow} \quad L_{E,\chi}(s) = \sum_{n \in \mathbb{N}} \frac{a_n \chi(n)}{n^s}.$$

By representation theory, there is a factorisation

$$L_{E/K}(s) = \prod_\chi L_{E,\chi}(s),$$

where $\chi : (\mathbb{Z}/m\mathbb{Z})^\times \to \mathbb{C}^\times$ runs over primitive Dirichlet characters.

A twisted BSD conjecture

Conjecture (Deligne–Gross)

The order of vanishing of $L_{E,\chi}(s)$ at $s = 1$ is equal to $\langle \chi, E(K)_{\mathbb{C}} \rangle$.

Unfortunately, a twisted version of strong BSD seems difficult.

Example (Dokchitser–Evans–Wiersema)

Let E_1 and E_2 be elliptic curves given by Cremona labels 307a1 and 307c1, and let $\chi : (\mathbb{Z}/11\mathbb{Z})^\times \to \mathbb{C}^\times$ be the primitive Dirichlet character of order 5 and conductor 11 given by $\chi(2) = \zeta_5$. Then

$$C_{E_i/K} = R_{E_i/K} = Ш_{E_i/K} = \mathrm{tor}_{E_i/K} = 1,$$

for $K \subseteq \mathbb{Q}(\zeta_{11})^+$, but

$$\mathscr{L}_{E_1,\chi} = 1, \qquad \mathscr{L}_{E_2,\chi} = \zeta_5(1 + \zeta_5^4)^2.$$

Algebraic twisted L-values

If $\mathrm{rk}_E = 0$, the **algebraic twisted L-value** is given by

$$\mathscr{L}_{E,\chi} := L_{E,\chi}(1) \cdot \frac{\tau(\overline{\chi})}{\Omega_E},$$

where $\tau(\overline{\chi})$ is the **Gauss sum**

$$\tau(\overline{\chi}) := \sum_{n \in (\mathbb{Z}/m\mathbb{Z})^\times} \overline{\chi}(n)\zeta_m^n.$$

In general $\mathscr{L}_{E,\chi} \in \overline{\mathbb{Q}}$, but some integrality statements are known.

Theorem (Wiersema–Wuthrich)

If E is semistable optimal of conductor N_E, and if χ is primitive of order k and conductor coprime to N_E, then $\mathscr{L}_{E,\chi} \in \mathbb{Z}[\zeta_k]$.

There are stronger statements under the *Manin constant conjecture*.

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Real algebraic twisted L-values

Assume that $\mathrm{rk}_E = 0$, and that χ is primitive of order $k > 2$.

Lemma (Kisilevsky–Nam)

Let ω_E be the "root number" of E. Then $\lambda_\chi \cdot \mathscr{L}_{E,\chi} \in \mathbb{Z}[\zeta_k]^+$, where

$$\lambda_\chi := \begin{cases} 1 & \text{if } \omega_E \cdot \chi(-N_E) = 1, \\ \chi(m) - \overline{\chi(m)} & \text{if } \omega_E \cdot \chi(-N_E) = -1, \\ 1 + \omega_E \cdot \overline{\chi(-N_E)} & \text{if } \omega_E \cdot \chi(-N_E) \neq \pm 1, \end{cases} \qquad m \in \mathbb{Z}.$$

This is called the **real algebraic twisted L-value** $\mathscr{L}^+_{E,\chi}$.

Example ($k = 3$)

$$\mathbb{Z} \ni \mathscr{L}^+_{E,\chi} = \begin{cases} \Re(\mathscr{L}_{E,\chi}) & \text{if } \omega_E \cdot \chi(N_E) = 1, \\ 2\Re(\mathscr{L}_{E,\chi}) & \text{if } \omega_E \cdot \chi(N_E) \neq 1. \end{cases}$$

Some observations

Let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ run over primes $p \equiv 1 \mod 3$.

Example (Kisilevsky–Nam)

Let E be the elliptic curve given by the Cremona label 11a1.

p	7	13	19	31	37	43	61	67	73	79	97
$\mathscr{L}^+_{E,\chi}$	5	−10	−10	5	20	5	−10	15	5	15	−30
$\overline{\mathscr{L}}^+_{E,\chi}$	1	−2	−2	1	4	1	−2	3	1	3	−6
$[\overline{\mathscr{L}}^+_{E,\chi}]_3$	1	1	1	1	1	1	1	0	1	0	0
$[\#E(\mathbb{F}_p)]_3$	1	1	2	1	2	2	2	0	1	0	0
$\chi(N_E)$	ζ_3	ζ_3	1	$\overline{\zeta_3}$	1	1	1	$\overline{\zeta_3}$	ζ_3	$\overline{\zeta_3}$	$\overline{\zeta_3}$

p	103	109	127	139	151	157	163	181	193	199
$\mathscr{L}^+_{E,\chi}$	30	5	15	5	0	0	80	50	−5	−55
$\overline{\mathscr{L}}^+_{E,\chi}$	6	1	3	1	0	0	16	10	−1	−11
$[\overline{\mathscr{L}}^+_{E,\chi}]_3$	0	1	0	1	0	0	1	1	2	1
$[\#E(\mathbb{F}_p)]_3$	0	1	0	1	0	0	1	1	1	2
$\chi(N_E)$	ζ_3	$\overline{\zeta_3}$	$\overline{\zeta_3}$	ζ_3	ζ_3	ζ_3	$\overline{\zeta_3}$	$\overline{\zeta_3}$	1	1

Here, $\overline{\mathscr{L}}^+_{E,\chi} := \mathscr{L}^+_{E,\chi} / \gcd_{\chi'}\{\mathscr{L}^+_{E,\chi'}\}$.

Some phenomena

If E is the elliptic curve given by the Cremona label 11a1,

$$\overline{\mathscr{L}}^{+}_{E,\chi} \equiv_3 \begin{cases} 0 & \text{if } \#E(\mathbb{F}_p) \equiv 0 \mod 3, \\ 2 & \text{if } \#E(\mathbb{F}_p) \equiv 1 \mod 3 \text{ and } \chi(N_E) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

KN computed $\overline{\mathscr{L}}^{+}_{E,\chi}$ modulo 3 for six elliptic curves.

- For 14a1, $\overline{\mathscr{L}}^{+}_{E,\chi} \equiv 2 \mod 3$ often occurs.
- For 11a1, 15a1, 17a1, $\overline{\mathscr{L}}^{+}_{E,\chi} \equiv 2 \mod 3$ rarely occurs.
- For 19a1, 37b1, $\overline{\mathscr{L}}^{+}_{E,\chi} \equiv 2 \mod 3$ never occurs.

Theorem (A.)

I can partially explain the DEW and KN phenomena.

The modularity theorem

Let E be a semistable optimal elliptic curve over $\mathbb{Q}$ of conductor N_E.

Theorem (Taylor–Wiles)

There is an eigenform $f_E \in S_2(\Gamma_0(N_E))$ with (Hecke) L-function $L_{f_E}(s) = L_E(s)$, such that the Hecke operator T_p has eigenvalue a_p.

For any cusp form $f \in S_k(\Gamma)$, its L-function is a Mellin transform

$$L_f(s) := \frac{(-2\pi i)^s}{\Gamma(s)} \int_0^\infty z^{s-1} f(z) \mathrm{d}z.$$

Set $s = 1$:

$$L_f(1) = -2\pi i \int_0^\infty f(z) \mathrm{d}z =: -\langle \{0, \infty\}, f \rangle.$$

This is a *period* of the *modular symbol* $\{0, \infty\}$.

Classical modular symbols

Let $\mathcal{H}$ be the extended upper half plane, and let $\phi : \mathcal{H} \twoheadrightarrow \mathcal{H}/\Gamma =: X_\Gamma$.

A **modular symbol** is a class $\{x, y\} \in H_1(X_\Gamma, \mathbb{R})$ for any $x, y \in \mathcal{H}$.

- If $\Gamma \cdot x = \Gamma \cdot y$, then $\phi(x \rightsquigarrow y) \in H_1(X_\Gamma, \mathbb{Z})$, and conversely any $\gamma \in H_1(X_\Gamma, \mathbb{Z})$ arises from $x, y \in \mathcal{H}$ in the same Γ-orbit. Define

$$\{x, y\} := \phi(x \rightsquigarrow y).$$

- The map $H_1(X_\Gamma, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma), \mathbb{C})$ given by $\gamma \mapsto \langle \gamma, \cdot \rangle$ extends to $\psi : H_1(X_\Gamma, \mathbb{R}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(S_2(\Gamma), \mathbb{C})$. Define

$$\{x, y\} := \psi^{-1} \langle \phi(x \rightsquigarrow y), \phi^*(\cdot) \rangle.$$

Note that

$$\{x, x\} = 0, \quad \{x, y\} = -\{y, x\}, \quad \{x, y\} + \{y, z\} = \{x, z\},$$

$$\langle \{x, y\}, M \cdot f \rangle = \langle \{M \cdot x, M \cdot y\}, f \rangle, \qquad M \in \Gamma.$$

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L-values as periods

Let $p \nmid N_E$. The Hecke operator T_p acts on $H_1(X_\Gamma, \mathbb{Q})$ by

$$T_p \cdot \{x, y\} = \{px, py\} + \sum_{n=0}^{p-1} \{\tfrac{x+n}{p}, \tfrac{y+n}{p}\}.$$

Lemma (Manin)

$$-\#E(\mathbb{F}_p) \cdot \mathscr{L}_E = \frac{1}{\Omega_E} \sum_{n=1}^{p-1} \langle \{0, \tfrac{n}{p}\}, f_E \rangle.$$

Proof.

Set $\{x, y\} = \{0, \infty\}$ in the Hecke action and apply the pairing $\langle \cdot, f_E \rangle$:

$$\underbrace{(1 + p - a_p)}_{\#E(\mathbb{F}_p)} \cdot \underbrace{\langle \{0, \infty\}, f_E \rangle}_{-L_E(1)} = \sum_{n=1}^{p-1} \underbrace{\langle \{0, \tfrac{n}{p}\}, f_E \rangle}_{???}.$$

Multiply by $\frac{1}{\Omega_E}$. □

Twisted L-values as periods

Lemma (Manin)

$$\mathscr{L}_{E,\chi} = \frac{1}{\Omega_E} \sum_{n=1}^{p-1} \overline{\chi}(n) \langle \{0, \tfrac{n}{p}\}, f_E \rangle.$$

Proof.

For any $m \in (\mathbb{Z}/p\mathbb{Z})^\times$, the discrete Fourier transform of $\chi(m)$ is

$$\chi(m) = \frac{1}{\tau(\overline{\chi})} \sum_{n=1}^{p-1} \overline{\chi}(n) \zeta_p^{mn}.$$

Substitute into $\sum_m a_m \chi(m) q^m$ and apply the Mellin transform:

$$L_{E,\chi}(1) = \frac{1}{\tau(\overline{\chi})} \sum_{n=1}^{p-1} \overline{\chi}(n) \underbrace{\langle \{0, \infty\}, M \cdot f_E \rangle}_{\text{apply properties}}, \qquad M := \begin{pmatrix} p & k \\ 0 & p \end{pmatrix}.$$

Multiply by $\frac{\tau(\overline{\chi})}{\Omega_E}$. □

A congruence for L-values

Let E be a semistable optimal elliptic curve over $\mathbb{Q}$ of conductor N_E, let $p \nmid N_E$, and let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character. Then

$$-\#E(\mathbb{F}_p) \cdot \mathscr{L}_E = \frac{1}{\Omega_E} \sum_{n=1}^{p-1} \langle \{0, \tfrac{n}{p}\}, f_E \rangle,$$

$$\mathscr{L}_{E,\chi} = \frac{1}{\Omega_E} \sum_{n=1}^{p-1} \overline{\chi}(n) \langle \{0, \tfrac{n}{p}\}, f_E \rangle.$$

Corollary (A)

If χ has prime order k, then

$$-\#E(\mathbb{F}_p) \cdot \mathscr{L}_E \equiv \mathscr{L}_{E,\chi} \mod (1 - \zeta_k).$$

Proof.

By integrality, the lemmata, and $\overline{\chi} \equiv 1 \mod (1 - \zeta_k)$. □

A congruence for cubic twists

Let $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}^\times$ be a cubic primitive Dirichlet character.

Corollary (B)

$$\mathscr{L}^+_{E,\chi} \equiv_3 \#E(\mathbb{F}_p) \cdot \mathscr{L}_E \cdot \begin{cases} 2 & \text{if } \omega_E \cdot \chi(N_E) = 1, \\ 1 & \text{if } \omega_E \cdot \chi(N_E) \neq 1. \end{cases}$$

Proof.

By cases of corollary (A). □

Corollary (C)

$$\overline{\mathscr{L}}^+_{E,\chi} \equiv_3 \#E(\mathbb{F}_p) \cdot \mathscr{L}_E \cdot \gcd_{\chi'}\{\mathscr{L}^+_{E,\chi'}\} \cdot \begin{cases} 2 & \text{if } \omega_E \cdot \chi(N_E) = 1, \\ 1 & \text{if } \omega_E \cdot \chi(N_E) \neq 1. \end{cases}$$

Proof.

By cases of corollary (B). □

DEW phenomena

Recall that if E_1 and E_2 are elliptic curves given by Cremona labels 307a1 and 307c1, and $\chi : (\mathbb{Z}/11\mathbb{Z})^\times \to \mathbb{C}^\times$ is the primitive Dirichlet character of order 5 and conductor 11 given by $\chi(2) = \zeta_5$, then

$$C_{E_i/K} = R_{E_i/K} = Ш_{E_i/K} = \mathrm{tor}_{E_i/K} = 1,$$

for $K \subseteq \mathbb{Q}(\zeta_{11})^+$, but

$$\mathscr{L}_{E_1,\chi} = 1, \qquad \mathscr{L}_{E_2,\chi} = \zeta_5(1+\zeta_5^4)^2.$$

Note that $\mathscr{L}_{E_i} = 1$, and

$$\#E_1(\mathbb{F}_{11}) = 9, \qquad \#E_2(\mathbb{F}_{11}) = 16,$$

so corollary (A) says $\mathscr{L}_{E_1,\chi} \not\equiv \mathscr{L}_{E_2,\chi} \mod (1-\zeta_5)$.

In fact, corollary (A) partially explains all examples in DEW where χ is quintic, and fully explains all examples in DEW where χ is cubic.

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Insufficiency of congruence

Unfortunately, there are elliptic curves E_1 and E_2 over $\mathbb{Q}$, where $\mathscr{L}_{E_1,\chi} \equiv \mathscr{L}_{E_2,\chi} \mod (1-\zeta_5)$, but $\mathscr{L}_{E_1,\chi} \neq \mathscr{L}_{E_2,\chi}$.

Example (A.)

Let E_1 and E_2 be elliptic curves given by Cremona labels 130b3 and 312c3, and let $\chi : (\mathbb{Z}/11\mathbb{Z})^\times \to \mathbb{C}^\times$ be the primitive Dirichlet character of order 5 and conductor 11 given by $\chi(2) = \zeta_5$. Then

$$C_{E_i/K} = 2, \qquad R_{E_i/K} = 1, \qquad Ш_{E_i/K} \cong \mathrm{tor}_{E_i/K} \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

for $K \subseteq \mathbb{Q}(\zeta_{11})^+$, and furthermore $\#E_i(\mathbb{F}_{11}) = 12$ and $\mathscr{L}_{E_i} = \frac{1}{2}$, but

$$\mathscr{L}_{E_1,\chi} = -4\zeta_5^3, \qquad \mathscr{L}_{E_2,\chi} = -4\zeta_5,$$

which are not equal but congruent modulo $(1-\zeta_5)$.

Heuristically, the norm of $\overline{\mathscr{L}}^+_{E,\chi}$ is the χ-component of $Ш_E$.

KN phenomena

Recall that if E is the elliptic curve given by the Cremona label 11a1,

$$\overline{\mathscr{L}}^{+}_{E,\chi} \equiv_3 \begin{cases} 0 & \text{if } \#E(\mathbb{F}_p) \equiv 0 \mod 3, \\ 2 & \text{if } \#E(\mathbb{F}_p) \equiv 1 \mod 3 \text{ and } \chi(N_E) = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Note that $\mathscr{L}_E \cdot \gcd_{\chi'}\{\mathscr{L}^{+}_{E,\chi'}\} = 1$ and $\omega_E = 1$, so corollary (C) says

$$\overline{\mathscr{L}}^{+}_{E,\chi} \equiv_3 \begin{cases} 2\#E(\mathbb{F}_p) & \text{if } \chi(N_E) = 1, \\ \#E(\mathbb{F}_p) & \text{if } \chi(N_E) \neq 1. \end{cases}$$

This fully explains the three cases, except for when $\#E(\mathbb{F}_p) \equiv 1 \mod 3$ or $\chi(N_E) = 1$. In fact, corollary (C) fully explains $\overline{\mathscr{L}}^{+}_{E,\chi}$ modulo 3 for any elliptic curve E over $\mathbb{Q}$ where

- E does not have rational 3-isogenies, and
- the 3-division field $\mathbb{Q}(x(E[3]))$ of E contains $\sqrt[3]{N_E}$.

The missing piece

Theorem (A.–Dokchitser)

Assume that

- *E does not have rational 3-isogenies, and*
- *the 3-division field $\mathbb{Q}(x(E[3]))$ of E contains $\sqrt[3]{N_E}$.*

If $\#E(\mathbb{F}_p) \equiv 2 \mod 3$, then $\chi(N_E) = 1$.

Proof.

The assumptions imply that

$$\mathrm{Gal}(\mathbb{Q}(x(E[3]))/\mathbb{Q}) \cong \mathrm{PGL}_2(\mathbb{F}_3) \cong S_4.$$

This has a quotient

$$\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{N_E}, \zeta_3)/\mathbb{Q}) \cong S_4/K_4 \cong S_3.$$

If $\#E(\mathbb{F}_p) \equiv 2 \mod 3$, then $a_p \equiv 0 \mod 3$, so $\phi_p^2 = 1$ in S_4. By group theory, $\phi_p = 1$ in S_3, but it acts as $\chi(N_E)$ on $\mathbb{Q}(\sqrt[3]{N_E}, \zeta_3)$. □

Other KN phenomena

Key idea: understand how ϕ_p acts on $\mathbb{Q}(x(E[3]))$ and $\mathbb{Q}(\sqrt[3]{N_E}, \zeta_3)$.

In general, $\phi_p \neq 1$ in $\mathrm{Gal}(\mathbb{Q}(x(E[3]))/\mathbb{Q}) \leq \mathrm{PGL}_2(\mathbb{F}_3)$.

- If E has rational 3-isogenies, then $\overline{\mathscr{L}}_{E,\chi}^{+}$ modulo 3 is partially explained by how ϕ_p acts on the 9-division field $\mathbb{Q}(x(E[9]))$.
- If $\mathbb{Q}(x(E[3]))$ does not contain $\sqrt[3]{N_E}$, then $\overline{\mathscr{L}}_{E,\chi}^{+}$ modulo 3 is fully explained by how ϕ_p acts on $\mathbb{Q}(x(E[3]), \sqrt[3]{N_E})$.

The elliptic curves given by Cremona labels 11a1, 15a1, 17a1 are generic, but those given by 14a1, 19a1, 37b1 are special.

Theorem (A.)

I understand how ϕ_p acts on $\mathbb{Q}(x(E[9]), \sqrt[3]{N_E})$.

This crucially uses the classification of 3-adic images of Galois for elliptic curves over $\mathbb{Q}$ by Rouse–Sutherland–Zureick-Brown.

Future work

Here are some potential extensions, listed in increasing difficulty:

- replace $\mathbb{Q}$ with a global field
- replace χ with an Artin representation
- replace E with the Jacobian of a higher genus curve
- remove the $\mathrm{rk}_E = 0$ assumption