London School of Geometry and Number Theory

London Junior Number Theory Seminar

The Euler system of Heegner points [1]

David Ang

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[1] Victor Kolyvagin, 1989. **Euler Systems**, in *Grothendieck Festschrift*

Overview

- Introduction
  - From Gross-Zagier to Kolyvagin
  - Application to BSD
  - The main result
- Generalised Selmer groups
  - Selmer structures
  - Application of Tate duality
  - Application of Chebotarev density
- The Euler system of Heegner points
  - Heegner points of higher conductors
  - Derived Kolyvagin classes
  - Computing the Selmer group

From Gross-Zagier to Kolyvagin

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- A **basic Heegner point**

$$P_K := \sum_{\sigma \in \mathrm{Gal}(K^1/K)} \sigma(P_1) \in E(K).$$

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Recall the Gross-Zagier formula.

Theorem (Gross-Zagier, 1986)

There is some $c \neq 0$ *such that* $L'(E/K, 1) = c \cdot \widehat{h}(P_K)$.

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This *almost* proves weak BSD for analytic rank ≤ 1!

Application to BSD

Theorem (Weak BSD for analytic rank ≤ 1)

Assume $\mathrm{ord}_{s=1}\, L(E/\mathbb{Q}, s) \leq 1$*. Then*

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Proof (for $\epsilon = -$).

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Thus $E(\mathbb{Q})_{/\mathrm{tors}} = \mathbb{Z} \cdot \frac{1}{n} P_K$, so $\mathrm{rk}_{\mathbb{Z}} E(\mathbb{Q}) = 1$. □

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Let $\ell \in \mathbb{N}$ *be an odd prime of good reduction such that*

$$\mathrm{Gal}(\mathbb{Q}(E[\ell])/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{F}_\ell), \qquad P_K \notin \ell E(K).$$

Then $\mathrm{Sel}(K, E[\ell]) = \mathbb{F}_\ell \cdot \delta(P_K)$.

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Proof (of Kolyvagin).

For any $\ell \in \mathbb{N}$, there is a short exact sequence

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By inflation-restriction, there is a short exact sequence

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Let $v \nmid \ell$ have good reduction. Then there is a short exact sequence

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A **Selmer structure** on M is an assignment

$$v \longmapsto H^1_f(K_v, M) \subseteq H^1(K_v, M),$$

such that $H^1_f(K_v, M) = H^1(G_v^{\mathrm{ur}}, M^{I_v})$ for almost all places v of K.

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- The **unramified** Selmer structure has

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- The **geometric** Selmer structure has

$$H^1_f(K_v, M) := E(K_v)/\ell E(K_v), \qquad H^1_s(K_v, M) := H^1(K_v, E)[\ell].$$

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Diagram chase. □

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There is a finite set S of inert primes of $K/\mathbb{Q}$ such that

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Idea: choose appropriate L'/L to bound $\mathrm{Sel}_S^{\pm}$.

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Both $\mathrm{Sel}^{S\pm}$ and $H^1_s(K_v, E[\ell])^\pm$ in

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Let $n = pq$. Then

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By E-S theory, $\phi(H_p D) = a_p \phi(D)$ for any $D \in \mathrm{Div}(X_0(N))$. $\square$

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0 & \longrightarrow & H^1_f(K, E[\ell])^{\epsilon_n} & \xrightarrow{\delta} & H^1(K, E[\ell])^{\epsilon_n} & \longrightarrow & H^1_s(K, E[\ell])^{\epsilon_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mathrm{res}_n & & \downarrow & & \\
0 & \to & H^1_f(K^n, E[\ell])^{G_n \epsilon_n} & \underset{\delta_n}{\to} & H^1(K^n, E[\ell])^{G_n \epsilon_n} & \to & H^1_s(K^n, E[\ell])^{G_n \epsilon_n} & & \\
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\end{array}.$$

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Derived Kolyvagin classes

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Thus $(\mathrm{res}_n(c(n))_v)^s = 0$ by exactness. □

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Compute Sel^{ϵ} and $\mathrm{Sel}^{-\epsilon}$ separately.

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Relaxed:

- Fact: each $H^1_s(K_p, E[\ell])^{\pm}$ is one-dimensional.
- Show $c(n) \in \mathrm{Sel}^{S\epsilon_n}$ is non-zero in $H^1_s(K_p, E[\ell])$ for some n.

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Thank you!