The group law on an elliptic curve ¹

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Postgraduate Seminar

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¹ Angdinata, David Kurniadi and Xu, Junyan. An Elementary Formal Proof of the Group Law on Weierstrass Elliptic Curves in Any Characteristic. Fourteenth International Conference on Interactive Theorem Proving (ITP 2023)□ → <

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Talk overview:

- What is an elliptic curve?
- Why is it a group?
- Where is the problem then?
- How did we do it?

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- The Atkin-Morain primality test and Lenstra's factorisation method use elliptic curves and are two of the fastest known algorithms.

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Theorem (long Weierstrass model)

Any elliptic curve E over F can be given by E(X, Y) = 0, where

$$E(X,Y) := Y^2 + a_1 XY + a_3 Y - (X^3 + a_2 X^2 + a_4 X + a_6),$$

for some $a_i \in F$ such that $\Delta \neq 0$, 2

$${}^{2}\Delta := -(a_{1}^{2}+4a_{2})^{2}(a_{1}^{2}a_{6}+4a_{2}a_{6}-a_{1}a_{3}a_{4}+a_{2}a_{3}^{2}-a_{4}^{2}) - 8(2a_{4}+a_{1}a_{3})^{3} - 27(a_{3}^{2}+4a_{6})^{2} + 9(\hat{a}_{1}^{2}+4a_{2})(2a_{4}+a_{1}a_{3}^{2})(a_{3}^{2}+4a_{6}) \xrightarrow{\circ} 0 < 0$$

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Proof.

Follows from the *Riemann-Roch theorem* in algebraic geometry.

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If char(F) \neq 2, 3, then E has a **short Weierstrass model**, where $E(X, Y) := Y^2 - (X^3 + aX + b),$

for some $a, b \in F$ such that $\Delta = -16(4a^3 + 27b^2) \neq 0$.

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Note that in the **coordinate ring** $F[E] := F[X, Y]/\langle E(X, Y) \rangle$,

$$-(Y \cdot \sigma(Y)) = Y^{2} + a_{1}XY + a_{3}Y \equiv X^{3} + a_{2}X^{2} + a_{4}X + a_{6},$$

which is a polynomial only in X.

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Addition can be given by $(x_1, y_1) + (x_2, y_2) := -(x_3, y_3)$. Here,

$$\lambda := \begin{cases} \frac{y_1 - y_2}{x_1 - x_2} & x_1 \neq x_2\\ \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{y_1 - \sigma(y_1)} & y_1 \neq \sigma(y_1) \\ \infty & \text{otherwise} \end{cases}$$
$$x_3 := \lambda^2 + a_1\lambda - a_2 - x_1 - x_2,$$
$$y_3 := \lambda(x_3 - x_1) + y_1.$$

One may attempt to prove the axioms directly.

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In the generic case, 3 checking that their X-coordinates are equal is an equality of polynomials with 26,082 terms.

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Automation in an interactive theorem prover enables manipulation of multivariate polynomials with at most 5,000 terms.

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Existing interactive theorem provers have used Pf 1 (Théry 2007) or Pf 4 (Bartzia–Strub 2014), both assuming the short Weierstrass model.

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Our proof sets G = Cl(F[E]), and

 ϕ is injective \iff an ideal of F[E] is not principal, which is just a statement in ring theory.

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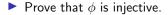
Example (of invertible fractional ideals)

Any nonzero ideal I such that $I \cdot J$ is principal for some ideal J.

Pf 5 (A.–Xu).

▶ Define a function $\phi : E(F) \rightarrow Cl(F[E])$.

> Prove that ϕ respects addition.

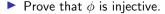


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Note that ϕ is well-defined since

$$\langle X - x, Y - y \rangle \cdot \langle X - x, Y - \sigma(y) \rangle = \langle X - x \rangle.$$

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• Prove that
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• Prove that ϕ respects addition. This holds since

$$\langle X - x_1, Y - y_1 \rangle \cdot \langle X - x_2, Y - y_2 \rangle \cdot \langle X - x_3, Y - \sigma(y_3) \rangle$$

= $\langle (Y - y_3) - \lambda (X - x_3) \rangle.$

Prove that ϕ is injective.

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Example (of norms)

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Combining both equalities and (\star) yields

$$\dim(F[E]/\langle f \rangle) \neq 1. \tag{(†)}$$

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Since dim(F) = 1, this contradicts (†)!

Conclusions

Some retrospectives:

- formalisation encouraged proof accessible to undergraduates
- novel injectivity proof and novel formalisation
- proof works for nonsingular points of Weierstrass curves
- heavy use of linear algebra and ring theory in Lean's mathlib
- generality of ideal class groups of integral domains
- plans for many more formalisation projects!

Thank you!