Twisted L-values of elliptic curves

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L-functions

Let E be an elliptic curve over $\mathbb{Q}$.

Recall that the L-function of E is

$$L(E,s) := \prod_p \frac{1}{\det(1 - p^{-s} \cdot \mathrm{Fr}_p^{-1} \mid \rho_{E,q}^{\vee I_p})}.$$

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Conjecture (Birch–Swinnerton-Dyer)

- *The order of vanishing r of $L(E,s)$ at $s=1$ is $\mathrm{rk}(E)$.*
- *The leading term of $L(E,s)$ at $s=1$ is*

$$\lim_{s\to 1} \frac{L(E,s)}{(s-1)^r} \cdot \frac{1}{\Omega(E)} = \frac{\mathrm{Reg}(E) \cdot \mathrm{Tam}(E) \cdot \#\text{Ш}(E)}{\#\mathrm{tor}(E)^2}.$$

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Let E be an elliptic curve over $\mathbb{Q}$. Let K be finite Galois over $\mathbb{Q}$.

Recall that the L-function of E/K is

$$L(E/K,s) := \prod_{\mathfrak{p}} \frac{1}{\det(1-\mathrm{Nm}(\mathfrak{p})^{-s}\cdot \mathrm{Fr}_{\mathfrak{p}}^{-1} \mid \rho_{E,q}^{\vee I_{\mathfrak{p}}})}.$$

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Twisted L-functions

Artin's formalism for L-functions gives

$$L(E/K, s) = \prod_{\rho:\mathrm{Gal}(K/\mathbb{Q})\to\mathbb{C}^\times} L(E, \rho, s)^{\dim \rho}.$$

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If K is abelian, then ρ corresponds to a Dirichlet character χ, and

$$L(E, s) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^s} \quad \overset{\chi}{\rightsquigarrow} \quad L(E, \chi, s) = \sum_{n\in\mathbb{N}} \frac{a_n \chi(n)}{n^s}.$$

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What can be said about $L(E,\rho,s)$ algebraically and analytically?

Algebraic result: twisted conjectures

Conjecture (Deligne–Gross)

The order of vanishing of $L(E, \rho, s)$ *at* $s = 1$ *is* $\langle \rho, E(K) \otimes_{\mathbb{Z}} \mathbb{C} \rangle$.

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What is the conjectural leading term? Assuming $L(E, 1) \neq 0$, define

$$\mathcal{L}(E, \chi) := L(E, \chi, 1) \cdot \frac{p}{\tau(\chi) \cdot \Omega(E)},$$

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Example (Dokchitser–Evans–Wiersema 2021)

Let E_1 and E_2 be the elliptic curves given by 1356d1 and 1356f1, and let χ be the cubic character of conductor 7 given by $\chi(3) = \zeta_3^2$. Then

$$\mathrm{Reg}(E_i/K) = \mathrm{Tam}(E_i/K) = Ш(E_i/K) = \mathrm{tor}(E_i/K) = 1,$$

for $K = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_7)^+$, but $\mathcal{L}(E_1, \chi) = \zeta_3^2$ and $\mathcal{L}(E_2, \chi) = -\zeta_3^2$.

Algebraic result: determining units

Assume E has conductor N and satisfies $c_1(E) = 1$, and assume χ has odd prime conductor $p \nmid N$ and odd prime order $q \nmid \#E(\mathbb{F}_p) \cdot \mathrm{BSD}(E)$.

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Theorem (Dokchitser–Evans–Wiersema 2021)

Let $\zeta := \chi(N)^{(q-1)/2}$. Then $\mathcal{L}(E, \chi) \cdot \zeta \in \mathbb{Z}[\zeta_q]^+ \setminus \{0\}$, and has norm

$$\mathrm{Nm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_q)^+}(\mathcal{L}(E, \chi) \cdot \zeta) = \pm \underbrace{\sqrt{\frac{\mathrm{BSD}(E/K)}{\mathrm{BSD}(E)}}}_{B},$$

where K is the degree q subfield of $\mathbb{Q}(\zeta_p)$ cut out by χ.

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Theorem (A. 2024)

If $q = 3$, *then*

$$\mathcal{L}(E, \chi) \cdot \zeta = \begin{cases} B & \text{if } \#E(\mathbb{F}_p) \cdot \mathrm{BSD}(E) \cdot B^{-1} \equiv 2 \mod 3 \\ -B & \text{if } \#E(\mathbb{F}_p) \cdot \mathrm{BSD}(E) \cdot B^{-1} \equiv 1 \mod 3 \end{cases}.$$

Analytic result: numerical evidence

Assume E as before, and let q be an odd prime. As p varies over odd primes $p \equiv 1 \mod q$, how does $\mathcal{L}(E, \chi)$ vary, for some uniform choice of primitive Dirichlet characters χ of conductor p and order q?

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Assume E as before, and let q be an odd prime. As p varies over odd primes $p \equiv 1 \mod q$, how does $\mathcal{L}(E,\chi)$ vary, for some uniform choice of primitive Dirichlet characters χ of conductor p and order q?

Example $(E = 20a1, q = 3)$

p	7	13	19	31	37	43	61	67	73	79
$\mathcal{L}(E,\chi)$	2	$-2\zeta_3$	-4	$-6\zeta_3$	$-6\zeta_3$	$6\zeta_3$	2	$-2\zeta_3$	0	$-6\zeta_3$

p	97	103	109	127	139	151	157	163	181
$\mathcal{L}(E,\chi)$	-4	$-6\zeta_3$	$6\zeta_3$	6	$18\zeta_3$	-4	$30\zeta_3$	$4\zeta_3$	$-2\zeta_3$

p	193	199	211	223	229	241	271	277	283
$\mathcal{L}(E,\chi)$	-4	$4\zeta_3$	$10\zeta_3$	$-24\zeta_3$	0	$-14\zeta_3$	$-6\zeta_3$	0	$6\zeta_3$

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Kisilevsky–Nam 2022 gave heuristic predictions on the asymptotic distribution of $\mathcal{L}(E,\chi)$, and computed data for the six elliptic curves given by 11a1, 14a1, 15a1, 17a1, 19a1, and 37b1.

Analytic result: residual densities

Let $X_{E,q}^{<n}$ be the set of order q primitive Dirichlet characters χ of conductor $p_\chi < n$ such that $\chi_1 \equiv \chi_2$ whenever $p_{\chi_1} = p_{\chi_2}$. Define

$$\delta_{E,q}(\lambda) := \lim_{n\to\infty} \frac{\#\{\chi \in X_{E,q}^{<n} \mid \mathcal{L}(E,\chi) \equiv \lambda \mod (1-\zeta_q)\}}{\#X_{E,q}^{<n}}.$$

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Theorem (A. 2024)

Let $m = 1 - \mathrm{ord}_q(\mathrm{BSD}(E))$. Then $\delta_{E,q}$ counts certain matrices in

$$G_{E,q^m} := \{M \in \mathrm{im}\overline{\rho_{E,q^m}} \mid \det(M) \equiv 1 \mod q\}.$$

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If $\overline{\rho_{E,q}}$ is surjective, then

$$\delta_{E,q}(\lambda) = \begin{cases} \frac{1}{q-1} & \text{if } L_0(q)L_4(q) = 1 \\ \frac{q}{q^2-1} & \text{if } L_0(q)L_4(q) = 0 \\ \frac{1}{q+1} & \text{if } L_0(q)L_4(q) = -1 \end{cases}, \qquad L_n(q) := \left(\frac{\frac{\lambda}{\mathrm{BSD}(E)} + n}{q} \right).$$

Analytic result: explicit algorithm

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If $q = 3$*, then* $\delta_{E,3}$ *only depends on* $\mathrm{im}\overline{\rho_{E,9}}$ *and* $b := 3\mathrm{BSD}(E) \bmod 9$.

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<table>
<tr><th>$\mathrm{im}\overline{\rho_{E,3}}$ or $\mathrm{im}\overline{\rho_{E,9}}$</th><th>$b$</th><th>$\delta_{E,3}(0)$</th><th>$\delta_{E,3}(1)$</th><th>$\delta_{E,3}(2)$</th><th>example</th></tr>
<tr><td rowspan="2">$\mathrm{GL}_2(\mathbb{F}_3)$</td><td>3</td><td>3/8</td><td>1/4</td><td>3/8</td><td>11a2</td></tr>
<tr><td>6</td><td>3/8</td><td>3/8</td><td>1/4</td><td>11a1</td></tr>
<tr><td rowspan="2">3B, 3Cs</td><td>3</td><td>1/2</td><td>0</td><td>1/2</td><td>50b3</td></tr>
<tr><td>6</td><td>1/2</td><td>1/2</td><td>0</td><td>50b1</td></tr>
<tr><td rowspan="2">3Nn</td><td>3</td><td>1/8</td><td>3/4</td><td>1/8</td><td>704e1</td></tr>
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<tr><td rowspan="2">3Ns</td><td>3</td><td>1/4</td><td>1/2</td><td>1/4</td><td>1690d1</td></tr>
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<tr><td>3.8.0.1</td><td>any</td><td>5/9</td><td>2/9</td><td>2/9</td><td>20a1</td></tr>
<tr><td rowspan="2">9.24.0.2,
9.72.0.(8,9,10),
27.648.18.1,
27.1944.55.(43,44)</td><td>1, 4, 7</td><td>1/3</td><td>2/3</td><td>0</td><td>108a1</td></tr>
<tr><td>2, 5, 8</td><td>1/3</td><td>0</td><td>2/3</td><td>36a1</td></tr>
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On the other hand, Birch's formula can be modified to give

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Scaling appropriately gives a $\mathbb{Z}[\zeta_q]$ congruence

$$-\mathcal{L}(E) \cdot \#E(\mathbb{F}_p) \equiv \mathcal{L}(E, \chi) \mod (1 - \zeta_q),$$

which proves the algebraic result.

Proof of analytic result

For the analytic result, note that $\mathcal{L}(E, \chi)$ varies according to

$$\#E(\mathbb{F}_p) = 1 + \det(\rho_{E,q}(\mathrm{Fr}_p)) - \mathrm{tr}(\rho_{E,q}(\mathrm{Fr}_p)) \mod q.$$

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Proof.

- Cancellation of torsion and Tamagawa numbers (Lorenzini 2011)
- Classification of $\mathrm{im}(\rho_{E,3})$ (Rouse–Sutherland–Zureick-Brown 2022)

□