Twisted L-values of elliptic curves

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L-functions

Let E be an elliptic curve over $\mathbb{Q}$. Let K be finite Galois over $\mathbb{Q}$.

Recall that the L-function of E/K is

$$L(E/K,s) := \prod_{\mathfrak{p}} \frac{1}{\det(1 - \mathrm{Nm}(\mathfrak{p})^{-s} \cdot \mathrm{Fr}_{\mathfrak{p}}^{-1} \mid \rho_{E,q}^{\vee I_{\mathfrak{p}}})}.$$

Conjecture (Birch–Swinnerton-Dyer)

- *The order of vanishing r of $L(E/K,s)$ at $s=1$ is* $\mathrm{rk}(E/K)$.
- *The leading term of $L(E/K,s)$ at $s=1$ is*

$$\underbrace{\lim_{s\to 1} \frac{L(E/K,s)}{(s-1)^r} \cdot \frac{\sqrt{\Delta(K)}}{\Omega(E/K)}}_{\mathscr{L}(E/K)} = \underbrace{\frac{\mathrm{Reg}(E/K)\cdot \mathrm{Tam}(E/K)\cdot \#\text{Ш}(E/K)}{\#\,\mathrm{tor}(E/K)^2}}_{\mathrm{BSD}(E/K)}.$$

Twisted L-functions

Artin's formalism for L-functions gives

$$L(E/K, s) = \prod_{\rho:\mathrm{Gal}(K/\mathbb{Q})\to\mathbb{C}^\times} L(E, \rho, s)^{\dim \rho}.$$

Here the L-function of E twisted by an Artin representation ρ is

$$L(E, \rho, s) := \prod_p \frac{1}{\det(1 - p^{-s} \cdot \mathrm{Fr}_p^{-1} \mid (\rho_{E,q}^\vee \otimes \rho^\vee)^{I_p})}.$$

If K is abelian, then ρ corresponds to a Dirichlet character χ, and

$$L(E, s) = \sum_{n\in\mathbb{N}} \frac{a_n}{n^s} \quad \overset{\chi}{\rightsquigarrow} \quad L(E, \chi, s) = \sum_{n\in\mathbb{N}} \frac{a_n \chi(n)}{n^s}.$$

What can be said about $L(E, \rho, s)$ algebraically and analytically?

Algebraic result: twisted conjectures

Conjecture (Deligne–Gross)

The order of vanishing of $L(E, \rho, s)$ at $s = 1$ is $\langle \rho, E(K) \otimes_{\mathbb{Z}} \mathbb{C} \rangle$.

What is the conjectural leading term? Assuming $L(E, 1) \neq 0$, define

$$\mathscr{L}(E, \chi) := L(E, \chi, 1) \cdot \frac{p}{\tau(\chi) \cdot \Omega(E)},$$

for any primitive Dirichlet character χ of conductor p.

Example (Dokchitser–Evans–Wiersema 2021)

Let E_1 and E_2 be the elliptic curves given by 1356d1 and 1356f1, and let χ be the cubic character of conductor 7 given by $\chi(3) = \zeta_3^2$. Then

$$\mathrm{Reg}(E_i/K) = \mathrm{Tam}(E_i/K) = \text{Ш}(E_i/K) = \mathrm{tor}(E_i/K) = 1,$$

for $K = \mathbb{Q}$ and $K = \mathbb{Q}(\zeta_7)^+$, but $\mathscr{L}(E_1, \chi) = \zeta_3^2$ and $\mathscr{L}(E_2, \chi) = -\zeta_3^2$.

Algebraic result: determining units

Assume E has conductor N and satisfies $c_1(E) = 1$, and assume χ has odd prime conductor $p \nmid N$ and odd prime order $q \nmid \#E(\mathbb{F}_p) \cdot \mathrm{BSD}(E)$.

Theorem (Dokchitser–Evans–Wiersema 2021)

Let $\zeta := \chi(N)^{(q-1)/2}$. Then $\mathscr{L}(E,\chi) \cdot \zeta \in \mathbb{Z}[\zeta_q]^+ \setminus \{0\}$, and has norm

$$\mathrm{Nm}_{\mathbb{Q}}^{\mathbb{Q}(\zeta_q)^+}(\mathscr{L}(E,\chi) \cdot \zeta) = \pm \underbrace{\sqrt{\frac{\mathrm{BSD}(E/K)}{\mathrm{BSD}(E)}}}_{B},$$

where K is the degree q subfield of $\mathbb{Q}(\zeta_p)$ cut out by χ.

Theorem (A. 2024)

If $q = 3$, then

$$\mathscr{L}(E,\chi) \cdot \zeta = \begin{cases} B & \text{if } \#E(\mathbb{F}_p) \cdot \mathrm{BSD}(E) \cdot B^{-1} \equiv 2 \mod 3, \\ -B & \text{if } \#E(\mathbb{F}_p) \cdot \mathrm{BSD}(E) \cdot B^{-1} \equiv 1 \mod 3. \end{cases}$$

Analytic result: numerical evidence

Assume E as before, and let q be an odd prime. As p varies over odd primes $p \equiv 1 \mod q$, how does $\mathscr{L}(E,\chi)$ vary, for some uniform choice of primitive Dirichlet characters χ of conductor p and order q?

Example ($E = 20a1, q = 3$)

p	7	13	19	31	37	43	61	67	73	79
$\mathscr{L}(E,\chi)$	2	$-2\zeta_3$	-4	$-6\zeta_3$	$-6\zeta_3$	$6\zeta_3$	2	$-2\zeta_3$	0	$-6\zeta_3$
mod 3	2	1	2	0	0	0	2	1	0	0

p	97	103	109	127	139	151	157	163	181
$\mathscr{L}(E,\chi)$	-4	$-6\zeta_3$	$6\zeta_3$	6	$18\zeta_3$	-4	$30\zeta_3$	$4\zeta_3$	$-2\zeta_3$
mod 3	2	0	0	0	0	2	0	1	1

p	193	199	211	223	229	241	271	277	283
$\mathscr{L}(E,\chi)$	-4	$4\zeta_3$	$10\zeta_3$	$-24\zeta_3$	0	$-14\zeta_3$	$-6\zeta_3$	0	$6\zeta_3$
mod 3	2	1	1	0	0	1	0	0	0

Kisilevsky–Nam 2022 gave heuristic predictions on the asymptotic distribution of $\mathscr{L}(E,\chi)$, and computed data for the six elliptic curves given by 11a1, 14a1, 15a1, 17a1, 19a1, and 37b1.

Analytic result: residual densities

Let $X_{E,q}^{<n}$ be the set of order q primitive Dirichlet characters χ of conductor $p_\chi < n$ such that $\chi_1 \equiv \chi_2$ whenever $p_{\chi_1} = p_{\chi_2}$. Define

$$\delta_{E,q}(\lambda) := \lim_{n\to\infty} \frac{\#\{\chi \in X_{E,q}^{<n} : \mathscr{L}(E,\chi) \equiv \lambda \mod (1-\zeta_q)\}}{\#X_{E,q}^{<n}}.$$

Theorem (A. 2024)

Let $m = 1 - \mathrm{ord}_q(\mathrm{BSD}(E))$. *Then* $\delta_{E,q}$ *counts certain matrices in*

$$G_{E,q^m} := \{M \in \mathrm{im}\,\overline{\rho_{E,q^m}} : \det(M) \equiv 1 \mod q\}.$$

If $\overline{\rho_{E,q}}$ *is surjective, then*

$$\delta_{E,q}(\lambda) = \begin{cases} \frac{1}{q-1} & \text{if } L_0(q)L_4(q) = 1, \\ \frac{q}{q^2-1} & \text{if } L_0(q)L_4(q) = 0, \\ \frac{1}{q+1} & \text{if } L_0(q)L_4(q) = -1, \end{cases} \qquad L_n(q) := \left(\frac{\frac{\lambda}{\mathrm{BSD}(E)} + n}{q}\right).$$

Analytic result: explicit algorithm

Theorem (A. 2024)

If $q = 3$, then $\delta_{E,3}$ only depends on $\mathrm{im}\,\overline{\rho_{E,9}}$ *and* $b := 3\,\mathrm{BSD}(E) \bmod 9$.

$\mathrm{im}\,\overline{\rho_{E,3}}$ or $\mathrm{im}\,\overline{\rho_{E,9}}$	b	$\delta_{E,3}(0)$	$\delta_{E,3}(1)$	$\delta_{E,3}(2)$	*example*
$\mathrm{GL}_2(\mathbb{F}_3)$	3	3/8	1/4	3/8	11*a*2
	6	3/8	3/8	1/4	11a1
3B, 3Cs	3	1/2	0	1/2	50b3
	6	1/2	1/2	0	50b1
3Nn	3	1/8	3/4	1/8	704e1
	6	1/8	1/8	3/4	245b1
3Ns	3	1/4	1/2	1/4	1690d1
	6	1/4	1/4	1/2	338d1
3.8.0.1	*any*	5/9	2/9	2/9	20a1
9.24.0.2, 9.72.0.(8,9,10), 27.648.18.1, 27.1944.55.(43,44)	1, 4, 7	1/3	2/3	0	108a1
	2, 5, 8	1/3	0	2/3	36a1
any		1	0	0	14*a*1

Proof of algebraic result

Manin's formalism for modular symbols compares $L(E,1)$ and $L(E,\chi,1)$.

The Hecke action on the space of modular symbols gives

$$-L(E,1)\cdot \#E(\mathbb{F}_p) = \sum_{a=1}^{p-1} \int_0^{\frac{a}{p}} 2\pi i f_E(z)\mathrm{d}z.$$

On the other hand, Birch's formula can be modified to give

$$L(E,\chi,1) = \frac{\tau(\chi)}{n} \sum_{a=1}^{p-1} \overline{\chi(a)} \int_0^{\frac{a}{p}} 2\pi i f_E(z)\mathrm{d}z.$$

Scaling appropriately gives a $\mathbb{Z}[\zeta_q]$ congruence

$$-\mathscr{L}(E)\cdot \#E(\mathbb{F}_p) \equiv \mathscr{L}(E,\chi) \mod (1-\zeta_q),$$

which proves the algebraic result.

Proof of analytic result

For the analytic result, note that $\mathscr{L}(E,\chi)$ varies according to

$$\#E(\mathbb{F}_p) = 1 + \det(\rho_{E,q}(\mathrm{Fr}_p)) - \mathrm{tr}(\rho_{E,q}(\mathrm{Fr}_p)) \mod q.$$

Chebotarev's density theorem says that $\rho_{E,q}(\mathrm{Fr}_p)$ varies uniformly in

$$G_{E,q^\infty} := \{M \in \mathrm{im}\,\rho_{E,q} : \det(M) \equiv 1 \mod q\}.$$

The following result reduces the computation from G_{E,q^∞} to G_{E,q^2}.

Theorem (A. 2024)

Let q be an odd prime. Then $\mathrm{ord}_q(\mathscr{L}(E)) \geq -1$.

Proof.

- Cancellation of torsion and Tamagawa numbers (Lorenzini 2011)
- Classification of $\mathrm{im}(\rho_{E,3})$ (Rouse–Sutherland–Zureick-Brown 2022)

□