Torsion subgroups of elliptic curves in Lean

SMS Spring Meeting: Formalization and Proof Assistants

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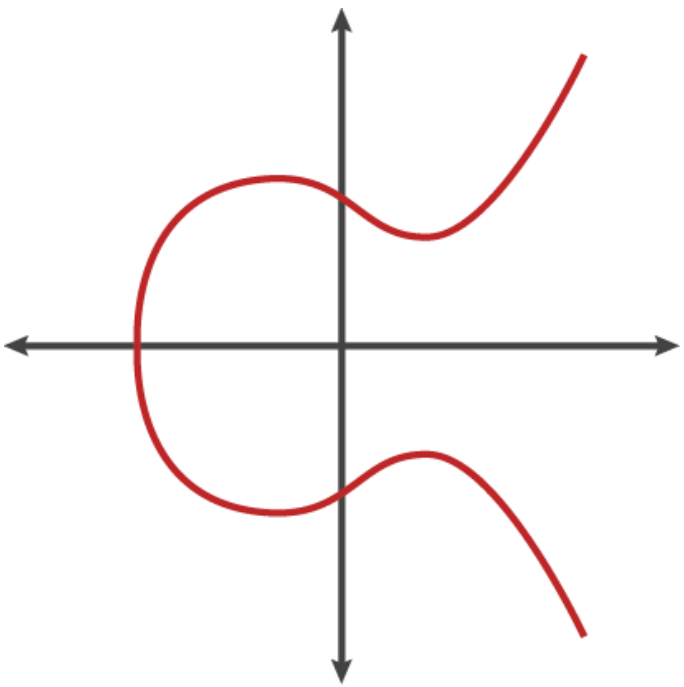
Elliptic curves

Elliptic curves are algebraic curves given by cubic equations, whose set of points can be endowed with the structure of an abelian group.



- Wiles's proof of Fermat's last theorem is a correspondence between elliptic curves and certain modular forms.
- The Birch and Swinnerton-Dyer conjecture predicts the rank of an elliptic curves in terms of L-functions.
- Intractability of the discrete logarithm problem forms the basis behind many public key cryptographic protocols.
- The Atkin–Morain primality test and Lenstra's factorisation method are two of the fastest known algorithms.

Weierstrass equations

Formally, an **elliptic curve** E **over a field** F is a smooth projective curve of genus one over F with a distinguished point O defined over F.

By the Riemann–Roch theorem, its set of points $E(F)$ is the zero locus of

$$f := Y^2 + a_1XY + a_3Y - (X^3 + a_2X^2 + a_4X + a_6),$$

for some $a_i \in F$ such that $\Delta \neq 0$, [1] with O being the point at infinity.

More specifically, there is an equivalence of categories

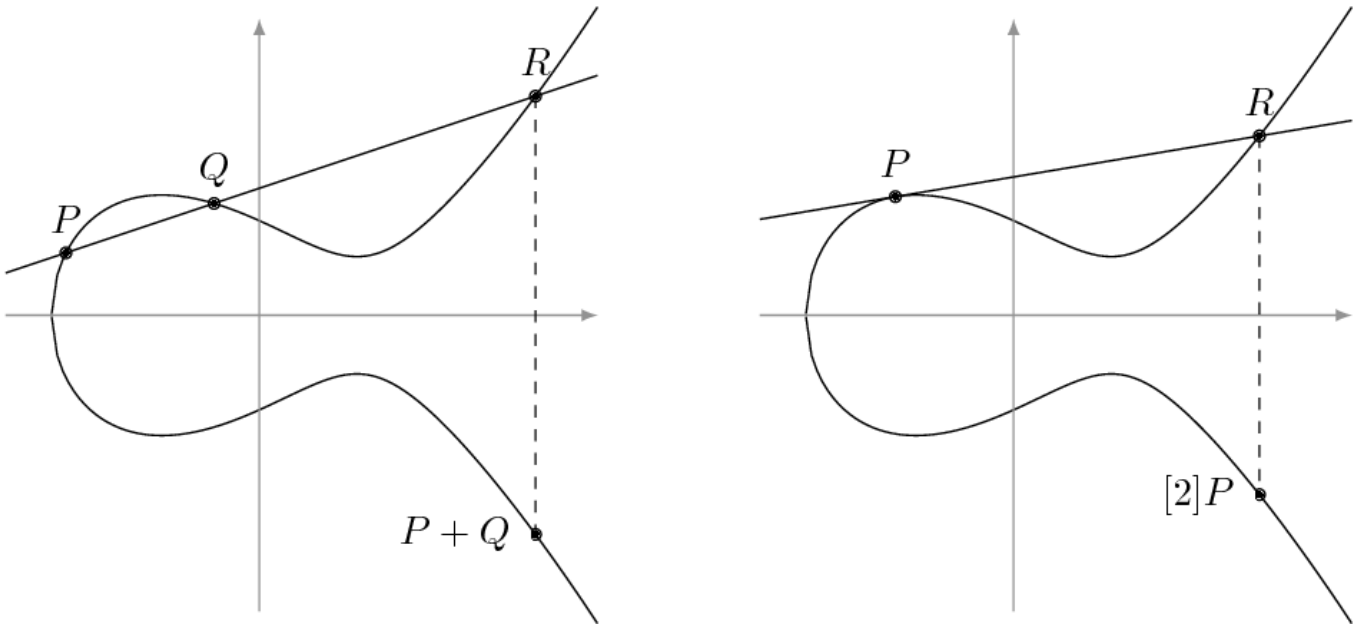
$$\{\text{elliptic curves over } F\} \cong \{(a_1, a_2, a_3, a_4, a_6) \in F^5 \text{ such that } \Delta \neq 0\}.$$

In `mathlib`, an **elliptic curve** E **over a ring** R is the data of a tuple $(a_1, a_2, a_3, a_4, a_6) \in R^5$ and a proof that $\Delta \in R^\times$. A **point** in $E(F)$ is either O or an **affine point** $(x, y) \in R^2$ such that $f(x, y) = 0$.

[1] $\Delta := -(a_1^2+4a_2)^2(a_1^2a_6+4a_2a_6-a_1a_3a_4+a_2a_3^2-a_4^2)-8(2a_4+a_1a_3)^3-27(a_3^2+4a_6)^2+9(a_1^2+4a_2)(2a_4+a_1a_3)(a_3^2+4a_6)$

Group law

There is an **addition law** of points in $E(F)$ defined geometrically.



In `mathlib`, the addition law is given by explicit rational functions.

Formalisation (A.–Xu, 2022)

This addition law makes $E(F)$ an abelian group with identity O.

The n-torsion subgroup

For any $P \in E(F)$ and any $n \in \mathbb{Z}$, denote multiplication-by-n by

$$[n](P) := \underbrace{P + \cdots + P}_{n},$$

and define the n-**torsion subgroup** by $E(F)[n] := \ker[n]$.

Formalisation (A.–Xu, 2024)

If the characteristic of F does not divide n, then $\#E(\overline{F})[n] = n^2$.

Exercise (3.7(d) in *The Arithmetic of Elliptic Curves* by Silverman)

Prove that for any affine point $(x, y) \in E(F)$,

$$[n](x, y) = \left(\frac{\phi_n(x, y)}{\psi_n(x, y)^2}, \frac{\omega_n(x, y)}{\psi_n(x, y)^3} \right).$$

Division polynomials

Assume $a_1 = a_2 = a_3 = 0$. Then $\phi_n, \omega_n, \psi_n \in F[X, Y]$ are defined by

$$\begin{aligned}
\psi_0 &:= 0,\\
\psi_1 &:= 1,\\
\psi_2 &:= 2Y,\\
\psi_3 &:= 3X^4 + 6a_4X^2 + 12a_6X - a_4^2,\\
\psi_4 &:= \psi_2 \cdot (2X^6 + 10a_4X^4 + 40a_6X^3\\
&\qquad - 10a_4^2X^2 - 8a_4a_6X - 4a_6^2 - 2a_4^3),
\end{aligned}$$

$$\begin{aligned}
\psi_{2n+1} &:= \psi_{n+2}\psi_n^3 - \psi_{n-1}\psi_{n+1}^3,\\
\psi_{2n} &:= \frac{\psi_{n-1}^2\psi_n\psi_{n+2} - \psi_{n-2}\psi_n\psi_{n+1}^2}{\psi_2},\\
\psi_{-n} &:= -\psi_n,\\
\phi_n &:= X\psi_n^2 - \psi_{n+1}\psi_{n-1},\\
\omega_n &:= \frac{\psi_{2n}}{2\psi_n}.
\end{aligned}$$

Observe that ϕ_n and ψ_n^2 can be viewed as polynomials in $F[X]$ modulo f.

Exercise (3.7(b) in *The Arithmetic of Elliptic Curves* by Silverman)

Prove that $\phi_n = X^{n^2} + \dots$ *and* $\psi_n^2 = n^2X^{n^2-1} + \dots$.

Arithmetic structure

Theorem (Mordell–Weil, 1929)

If K is a global field, then $E(K)$ is finitely generated.

By the structure theorem for finitely generated abelian groups,

$$E(K) \cong \mathrm{tor}(E/K) \oplus \mathbb{Z}^{\mathrm{rk}(E/K)},$$

where $\mathrm{rk}(E/K) \in \mathbb{N}$ is the **rank**, and the finite **torsion subgroup** is

$$\mathrm{tor}(E/K) := \bigcup_{n \in \mathbb{Z} \setminus \{0\}} E(K)[n] \leq E(K).$$

The proof of Mordell–Weil has two parts:

- Weak Mordell–Weil: $E(K)/nE(K)$ is finite for any $n \in \mathbb{Z} \setminus \{0\}$.
- Heights and descent: there is a height function on $E(K)$.

Michael Stoll formalised the latter a month ago.

The torsion subgroup

While $\mathrm{rk}(E/K)$ is mysterious, $\mathrm{tor}(E/K)$ is well-understood.

Theorem (Levin, 1968)

If K is the function field of a curve of genus g over a finite field, then $\#\,\mathrm{tor}(E/K)$ is bounded in terms of g.

Theorem (Merel, 1996)

If K is a number field, then $\#\,\mathrm{tor}(E/K)$ is bounded in terms of $[K:\mathbb{Q}]$.

Theorem (Mazur, 1978)

$$\mathrm{tor}(E/\mathbb{Q}) \cong \begin{cases} C_n, & \text{for } n = 1, \ldots, 10, 12, \\ C_2 \oplus C_{2n}, & \text{for } n = 1, \ldots, 4. \end{cases}$$

Computing torsion subgroups

Theorem (Lutz–Nagell, 1937)

Let E be an elliptic curve over $\mathbb{Q}$ with $a_1 = a_2 = a_3 = 0$ and $a_4, a_6 \in \mathbb{Z}$. If $(x, y) \in \mathrm{tor}(E/\mathbb{Q})$, then

1. *$x, y \in \mathbb{Z}$, and*
2. *either $y = 0$ or $y^2 \mid 4a_4^3 + 27a_6^2$.*

Note that $\Delta = -16(4a_4^3 + 27a_6^2)$, and

$$4a_4^3 + 27a_6^2 = 3(3y^2 + 4a_4x + 6a_6)y^2 - (3x^2 + 4a_4)\psi_3(x, y).$$

To compute $\mathrm{tor}(E/\mathbb{Q})$:

- Find all $y \in \mathbb{Z}$ such that $y^2 \mid 4a_4^3 + 27a_6^2$.
- For each such y, find all $x \in \mathbb{Z}$ such that $x^3 + a_4x + (a_6 - y^2) = 0$.
- For each such (x, y), check that it is in $\mathrm{tor}(E/\mathbb{Q})$.

An illustrative example

Let E be the elliptic curve $y^2 = x^3 + 8$ over $\mathbb{Q}$ and $(x, y) \in \mathrm{tor}(E/\mathbb{Q})$. Then $x, y \in \mathbb{Z}$, and either $y = 0$ or $y^2 \mid 1728 = 2^6 \cdot 3^3$. Hence,

$$y^2 \in \{0, 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 96, 108, 144, 192, 216, 288, 432, 576, 864, 1728\}.$$

y^2	0	1	4	9	16	36	64	144	576
$8 - y^2$	8	7	4	-1	-8	-28	-56	-136	-568
x	-2	N/A	N/A	1	2	N/A	N/A	N/A	N/A
y	0	± 1	± 2	± 3	± 4	± 6	± 8	± 12	± 24

Therefore, $(x, y) \in \{(-2, 0), (1, \pm 3), (2, \pm 4)\}$. Now,

$$[2](-2, 0) = O, \quad [2](1, \pm 3) = (-\tfrac{7}{4}, \mp \tfrac{13}{8}), \quad [2](2, \pm 4) = (-\tfrac{7}{4}, \pm \tfrac{13}{8}).$$

Thus, $\mathrm{tor}(E/\mathbb{Q}) = \{O, (-2, 0)\}$. In fact, $\mathrm{tor}(E/\mathbb{Q}) = \langle (-2, 0) \rangle \cong C_2$.

An elementary proof

Let $(x, y) \in \mathrm{tor}(E/\mathbb{Q})$ such that $y^2 = x^3 + a_4x + a_6$. May write

$$x = \frac{b}{d^2}, \qquad y = \frac{c}{d^3}, \qquad (b, d) = (c, d) = 1.$$

If $[n](x, y) \in \mathbb{Z}^2$, then $(x, y) \in \mathbb{Z}^2$, since the X-coordinate of $[n](x, y)$ is

$$\frac{\phi_n(x, y)}{\psi_n(x, y)^2} = \frac{x^{n^2} + \dots}{n^2 x^{n^2-1} + \dots} = \frac{b^{n^2} + d^2(\dots)}{d^2(n^2 b^{n^2-1} + \dots)}.$$

May assume that $[p](x, y) = O$ for $p > 2$. Then $\psi_p(x, y) = 0$, so

$$0 = \psi_p(x, y)^2 = p^2 x^{p^2-1} + \dots = \frac{p^2 b^{p^2-1} + d^2(\dots)}{d^{2(p^2-1)}}.$$

Thus $(x, y) \in \mathbb{Z}^2$. Now the X-coordinate of $[2](x, y) \in \mathrm{tor}(E/\mathbb{Q})$ is $x - \psi_3(x, y)/4y^2$, so $y^2 \mid \psi_3(x, y)$, and hence $y^2 \mid 4a_4^3 + 27a_6^2$.

Lean formalisation

Chris, ChatGPT, and Claude Code (CCCC) formalised this in a day!

Theorem (CCCC, 2026)

Let E be an elliptic curve over a number field K of class number one with $a_1 = a_3 = 0$ and $a_2, a_4, a_6 \in \mathcal{O}_K$. If $(x, y) \in \mathrm{tor}(E/K)$, and the primes dividing the order of (x, y) do not ramify in K, then

- $x, y \in \mathcal{O}_K$, *and*
- *either* $y = 0$ *or* $y^2 \mid (4a_4^3 + 27a_6^2) + a_2(4a_2^2 a_6 - a_2 a_4^2 - 18a_4 a_6)$.

When $a_1 \neq 0$ or $a_3 \neq 0$, they proved an alternative statement with $y^2 \mid \Delta$.

I am automating the computations of $\mathrm{tor}(E/\mathbb{Q})$ using `simprocs`.

When $4a_4^3 + 27a_6^2$ is massive, its factorisation can be extracted from the LMFDB to circumvent divisor `dsimprocs`, which uses trial division.