Tate's thesis [1] and epsilon factors

Galois representations and root numbers

David Kurniadi Angdinata

University College London

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[1] Tate (1950) *Fourier analysis in number fields and Hecke's zeta-functions*

Overview

Consider the Riemann ζ-function

$$\zeta(s) := \sum_{n\in\mathbb{N}^+} \frac{1}{n^s}.$$

Theorem (Riemann (1859))

$\zeta(s)$ has an analytic continuation to $\mathbb{C}$ with simple poles at $s = 0, 1$ and satisfies a functional equation $Z(s) = Z(1-s)$ where

$$Z(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\cdot\zeta(s).$$

Sketch of proof.

Write $Z(s)$ as a real integral of the theta series $\Theta(z) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 z}$.
The Poisson summation formula for $\mathbb{Z} \subset \mathbb{R}$ relates $\Theta(z)$ and $\Theta(1/z)$. $\square$

Can you generalise this to a number field K?

Overview

Consider the Dedekind ζ-function

$$\zeta_K(s) := \sum_{0 \neq I \trianglelefteq \mathcal{O}_K} \frac{1}{\mathrm{Nm}(I)^s}.$$

Theorem (Hecke (1917))

$\zeta_K(s)$ has an analytic continuation to $\mathbb{C}$ with simple poles at $s = 0, 1$ and satisfies a functional equation $Z_K(s) = Z_K(1-s)$ where

$$Z_K(s) := |\Delta_K|^{\frac{s}{2}} \cdot \left(\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\right)^{r_1} \cdot \left(2(2\pi)^{-s}\Gamma(s)\right)^{r_2} \cdot \zeta_K(s).$$

Sketch of proof.

Write $Z_K(s)$ as a real integral of a generalised theta series $\Theta_K(s)$ and apply the Poisson summation formula for a lattice in $\mathbb{R}^n$. $\square$

Can you explain the Γ-factors in the functional equation? Can you generalise this to L-functions $L(\chi, s)$ twisted by characters?

Overview

Tate (1950) answered both questions by giving a different proof of this. Idea: lift $\zeta_K(s)$ or $L(\chi, s)$ into global ζ-integrals over the locally compact topological group of idèles $\mathbb{A}_K^\times$ and apply techniques of Fourier analysis.

Note that there is an Euler product

$$Z_K(s) = |\Delta_K|^{\frac{s}{2}} \cdot \left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{r_1} \cdot \left(2(2\pi)^{-s}\Gamma(s)\right)^{r_2} \cdot \prod_{v \in V_K^f} \left(\sum_{n=0}^{\infty} q_v^{-ns}\right),$$

where V_K^f is the set of primes of K. On the other hand,

$$\mathbb{A}_K^\times = (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \times \overline{\prod_{v \in V_K^f}} K_v^\times.$$

Idea: the global ζ-integral over $\mathbb{A}_K^\times$ is the product of local ζ-integrals over $K_v^\times$, and the Γ-factors are local ζ-integrals at the archimedean places.

Local theory — Fourier analysis

Let F be a completion of a number field K_v, so $F/\mathbb{R}$ or $F/\mathbb{Q}_p$.

For $F = \mathbb{R}$, the Fourier transform

$$\widehat{f}(y) = \int_{-\infty}^{\infty} e^{-2\pi ixy} f(x) \mathrm{d}x$$

has three components. These are

- the integrable function f,
- the Lebesgue measure $\mathrm{d}x$, and
- the additive factor $e^{-2\pi ixy}$.

Each of these can be generalised for $F = \mathbb{C}$ and $F/\mathbb{Q}_p$.

Local theory — Haar measures

A locally compact topological group G can be endowed with a translation-invariant **Haar measure** $\mu_G = \int \mathrm{d}_G x$ unique up to scaling.

Examples

- Let $\mathrm{d}_{\mathbb{R}} x := \mathrm{d}x$ be the Lebesgue measure, and let $\mathrm{d}_{\mathbb{R}^\times} x := \mathrm{d}_{\mathbb{R}} x / |x|_{\mathbb{R}}$.
- Let $\mathrm{d}_{\mathbb{C}}(x+iy) := 2\mathrm{d}x\mathrm{d}y$ be twice the Lebesgue measure, and let $\mathrm{d}_{\mathbb{C}^\times} z := \mathrm{d}_{\mathbb{C}} z / |z|_{\mathbb{C}}$.
- Normalise $\mathrm{d}_{\mathbb{Q}_p} x$ such that $\mu_{\mathbb{Q}_p}(\mathbb{Z}_p) := 1$, so that

  $$\mu_{\mathbb{Q}_p}(a + p^n\mathbb{Z}_p) = \mu_{\mathbb{Q}_p}(p^n\mathbb{Z}_p) = p^{-n}\mu_{\mathbb{Q}_p}(\mathbb{Z}_p) = p^{-n},$$

  for all $a \in \mathbb{Q}_p$, and let

  $$\mathrm{d}_{\mathbb{Q}_p^\times} x := \frac{1}{1-p^{-1}} \frac{\mathrm{d}_{\mathbb{Q}_p} x}{|x|_v},$$

  so that $\mu_{\mathbb{Q}_p^\times}(\mathbb{Z}_p^\times) = 1$. If $G/\mathbb{Q}_p$, then μ_G and $\mu_{G^\times}$ should account for the valuation δ_v of the different ideal $\mathcal{D}_{G/\mathbb{Q}_p} \trianglelefteq \mathcal{O}_G$.

Local theory — Schwartz–Bruhat functions

What do you integrate over $F^\times$? **Schwartz–Bruhat** functions $F \to \mathbb{C}$.

- If $F = \mathbb{R}$, this is a function such that for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$,

$$\lim_{|x|\to\infty}\left(|x|^n \left|\frac{\mathrm{d}^m f}{\mathrm{d}x^m}\right|\right) = 0.$$

Example

Let $f(x) = f_0(x) := e^{-\pi x^2}$. Then

$$\begin{aligned}
\int_{\mathbb{R}^\times} f(x)|x|_{\mathbb{R}}^s \mathrm{d}_{\mathbb{R}^\times} x &= 2\int_0^\infty e^{-\pi x^2} x^s \ \frac{\mathrm{d}x}{x} \\
&= \int_0^\infty e^{-y}\left(\frac{y}{\pi}\right)^{\frac{s}{2}} \ \frac{\mathrm{d}y}{y} \qquad\qquad y = \pi x^2 \\
&= \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \\
&=: \Gamma_{\mathbb{R}}(s).
\end{aligned}$$

Local theory — Schwartz–Bruhat functions

What do you integrate over $F^\times$? **Schwartz–Bruhat** functions $F \to \mathbb{C}$.

- If $F = \mathbb{C}$, this is a function such that for all $n \in \mathbb{N}$ and $m_1, m_2 \in \mathbb{N}$,

$$\lim_{|x+iy|\to\infty} \left(|x+iy|_{\mathbb{C}}^n \left| \frac{\partial^{m_1+m_2} f}{\partial x^{m_1} \partial y^{m_2}} \right|_{\mathbb{C}} \right) = 0.$$

Example

Let $f(z) = f_0(z) := \frac{1}{\pi} e^{-2\pi z \bar{z}}$. Then

$$\begin{aligned} \int_{\mathbb{C}^\times} f(z) |z|_{\mathbb{C}}^s \mathrm{d}_{\mathbb{C}^\times} z &= \dots \\ &= 2(2\pi)^{-s}\Gamma(s) \\ &=: \Gamma_{\mathbb{C}}(s). \end{aligned}$$

Local theory — Schwartz–Bruhat functions

What do you integrate over $F^\times$? **Schwartz–Bruhat** functions $F \to \mathbb{C}$.

- If $F = K_v/\mathbb{Q}_p$, this is a linear combination of characteristic functions

$$\mathbb{I}_{a+\pi_v^n\mathcal{O}_v}(x) = \begin{cases} 1 & \text{if } x \in a + \pi_v^n\mathcal{O}_v, \\ 0 & \text{if } x \notin a + \pi_v^n\mathcal{O}_v, \end{cases}$$

Example

Let $f(x) = f_0(x) := \mathbb{I}_{\mathbb{Z}_p}(x)$. Then

$$\int_{\mathbb{Q}_p^\times} f(x)|x|_p^s \mathrm{d}_{\mathbb{Q}_p^\times} x = \frac{1}{1-p^{-1}} \int_{\mathbb{Z}_p} |x|_p^s \frac{\mathrm{d}_{\mathbb{Q}_p} x}{|x|_p}$$

$$= \sum_{n=0}^{\infty} \frac{p^{n-ns}}{1-p^{-1}} \int_{p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p} \mathrm{d}_{\mathbb{Q}_p} x = \sum_{n=0}^{\infty} p^{-ns}.$$

If $F/\mathbb{Q}_p$, let $f_0(x) := \mathbb{I}_{\mathcal{O}_F}(x)$ instead.

Local theory — additive characters

A Schwartz–Bruhat function $f : F \to \mathbb{C}$ has a Fourier transform

$$\widehat{f}(y) := \int_F \psi_F(xy) f(x) \mathrm{d}_F x,$$

where $\psi_F : F \to \mathbb{C}$ is an **additive character**.

- If $F = \mathbb{R}$, then $\psi_{\mathbb{R}}(x) := e^{-2\pi i x}$.
- If $F = \mathbb{C}$, then $\psi_{\mathbb{C}}(z) := e^{-2\pi i (z + \overline{z})}$.
- If $F = \mathbb{Q}_p$, then $\psi_{\mathbb{Q}_p}(x) := e^{2\pi i y}$, where $y \in \mathbb{Z}[p^{-1}]$ is such that $x \in y + \mathbb{Z}_p$. If $F/\mathbb{Q}_p$, apply the trace $\mathrm{tr} : F \to \mathbb{Q}_p$ first.

These are defined in such a way so that the Fourier inversion formula $\widehat{\widehat{f}}(x) = f(-x)$ holds, giving a duality between ψ_F and $\mathrm{d}_F x$. Indeed $\widehat{\widehat{f_0}} = f_0$, which is necessary in the Poisson summation formula.

Local theory — ζ-integrals

Let $f : F \to \mathbb{C}$ be a Schwartz–Bruhat function, and let $\chi : F^\times \to \mathbb{C}^\times$ be a multiplicative character. The **local ζ-integral** is defined to be

$$\zeta_F(f, \chi) := \int_{F^\times} f(x)\chi(x)\mathrm{d}_{F^\times} x,$$

which is independent of the dual pair $(\psi_F, \mathrm{d}_F x)$.

Theorem (Functional equation for the local ζ-integral)

There is a meromorphic function $L_F : \mathrm{Hom}(F^\times, \mathbb{C}^\times) \to \mathbb{C}^\times$ and a holomorphic function $\epsilon_F : \mathrm{Hom}(F^\times, \mathbb{C}^\times) \to \mathbb{C}^\times$ such that

$$\frac{\zeta_F(\widehat{f}, \chi^{-1}|\cdot|_F)}{L_F(\chi^{-1}|\cdot|_F)} = \epsilon_F(\chi)\frac{\zeta_F(f, \chi)}{L_F(\chi)}.$$

Here $L_F(\chi)$ is the **local L-factor** and $\epsilon_F(\chi)$ is the **local ϵ-factor**, which are both independent of the choice of f. The **local root number** is then defined to be $w_F(\chi) := \epsilon_F(\chi)/|\epsilon_F(\chi)| \in U(1)$.

Local theory — ϵ-factors

Determine multiplicative characters $\chi : F^\times \to \mathbb{C}^\times$ completely.

- Let $F = \mathbb{R}$. Then

$$\chi(x) = \eta(x)|x|_{\mathbb{R}}^s, \qquad \eta \in \{1, \mathrm{sgn}\}.$$

  - If $\eta = 1$, set $f(x) := f_0(x) = e^{-\pi x^2}$ and $L_{\mathbb{R}}(\chi) := \Gamma_{\mathbb{R}}(s)$. Then compute $\epsilon_{\mathbb{R}}(\chi) = 1$.
  - If $\eta = \mathrm{sgn}$, set $f(x) := xe^{-\pi x^2}$ and $L_{\mathbb{R}}(\chi) := \Gamma_{\mathbb{R}}(s+1)$. Then compute $\epsilon_{\mathbb{R}}(\chi) = -i$.

- Let $F = \mathbb{C}$. Then

$$\chi(z) = (z/\sqrt{z\bar{z}})^n |z|_{\mathbb{C}}^s, \qquad n \in \mathbb{Z}.$$

  - If $n = 0$, set $f(z) := f_0(z) = \frac{1}{\pi} e^{-2\pi z\bar{z}}$ and $L_{\mathbb{C}}(\chi) := \Gamma_{\mathbb{C}}(s)$. Then compute $\epsilon_{\mathbb{C}}(\chi) = 1$.
  - In general, set $f(z) := \frac{1}{\pi} z^n e^{-2\pi z\bar{z}}$ and $L_{\mathbb{C}}(\chi) := \Gamma_{\mathbb{C}}(s + \frac{1}{2}|n|)$. Then compute $\epsilon_{\mathbb{C}}(\chi) = i^{-|n|}$.

Local theory — ϵ-factors

Determine multiplicative characters $\chi : F^\times \to \mathbb{C}^\times$ completely.

- Let $F = K_v/\mathbb{Q}_p$. The **conductor** of χ is the least $n \in \mathbb{N}$ such that

$$\chi((1 + \pi_v^n \mathcal{O}_v) \cap \mathcal{O}_v^\times) = 1.$$

  If $n = 0$, then χ is said to be **unramified**.

  - If $n = 0$, set $f := \mathbb{I}_{\mathcal{O}_v}$ and $L_{K_v}(\chi) := (1 - \chi(\pi_v)^{-1})^{-1}$. Then compute

$$\epsilon_{K_v}(\chi) = q_v^{\frac{\delta_v}{2}} \chi(\pi_v)^{\delta_v}.$$

  - If $n > 0$, set $f := \mathbb{I}_{1+\pi_v^n \mathcal{O}_v}$ and $L_{K_v}(\chi) := 1$. Then compute

$$\epsilon_{K_v}(\chi) = \int_{K_v^\times} \psi_v(x) \chi(x)^{-1} \mathrm{d}_{K_v} x.$$

Local theory — ϵ-factors

Determine multiplicative characters $\chi : F^\times \to \mathbb{C}^\times$ completely.

F	χ	$L_F(\chi)$	$\epsilon_F(\chi)$
$\mathbb{R}$	$\lvert x\rvert_{\mathbb{R}}^s$	$\Gamma_{\mathbb{R}}(s)$	1
$\mathbb{R}$	$\operatorname{sgn}(x)\lvert x\rvert_{\mathbb{R}}^s$	$\Gamma_{\mathbb{R}}(s+1)$	$-i$
$\mathbb{C}$	$(z/\sqrt{z\bar{z}})^n\lvert z\rvert_{\mathbb{C}}^s$	$\Gamma_{\mathbb{C}}(s+\frac{1}{2}\lvert n\rvert)$	$i^{-\lvert n\rvert}$
K_v	unramified	$(1-\chi(\pi_v)^{-1})^{-1}$	$q_v^{\frac{\delta_v}{2}}\chi(\pi_v)^{\delta_v}$
K_v	ramified	1	$\int_{K_v^\times}\psi_v(x)\chi(x)^{-1}\mathrm{d}_{K_v}x$

Global theory — adèles and idèles

Let $V_K = V_K^f \cup V_K^\infty$ be the set of places of a number field K.

Consider the adèle ring

$$\mathbb{A}_K := \left\{ (x_v)_{v \in V_K} \in \prod_{v \in V_K} K_v : x_v \in \mathcal{O}_v \text{ for almost all } v \in V_K \right\}.$$

Its unit group is the idèle group

$$\mathbb{A}_K^\times := \left\{ (x_v)_{v \in V_K} \in \prod_{v \in V_K} K_v^\times : x_v \in \mathcal{O}_v^\times \text{ for almost all } v \in V_K \right\}.$$

Example

If $K = \mathbb{Q}$, then

$$\mathbb{A}_\mathbb{Q} \cong \mathbb{R} \times \bigcup_{n \in \mathbb{N}^+} \frac{1}{n} \prod_{p < \infty} \mathbb{Z}_p.$$

Global theory — adèles and idèles

Let $V_K = V_K^f \cup V_K^\infty$ be the set of places of a number field K.

The idèle group is endowed with the restricted product topology such that

$$\prod_{v \in S} U_v \times \prod_{v \in V_K \setminus S} \mathcal{O}_v^\times,$$

is an open basis for some finite $V_K^\infty \subseteq S \subset V_K$ and some open $U_v \subseteq K_v^\times$.

There is a diagonal embedding $K^\times \hookrightarrow \mathbb{A}_K^\times$. By the product formula,

$$|x|_{\mathbb{A}_K} := \prod_{v \in V_K} |x|_v = 1, \qquad x \in K^\times.$$

By Tychonoff's theorem, both the idèle group $\mathbb{A}_K^\times$ and the idèle class group $C_K := \mathbb{A}_K^\times / K^\times$ are locally compact topological groups.

Global theory — Hecke characters

A **Hecke character** is a character of the idèle class group, that is a continuous homomorphism $C_K \to \mathbb{C}^\times$ with the discrete topology on $\mathbb{C}^\times$.

Examples

- A Dirichlet character $\phi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ induces a Hecke character

$$C_\mathbb{Q} \cong \mathbb{R}^+ \times \prod_{p<\infty} \mathbb{Z}_p^\times \twoheadrightarrow \prod_{p|n} (\mathbb{Z}_p/n\mathbb{Z}_p)^\times \cong (\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\phi} \mathbb{C}^\times$$

  of finite order. Indeed, Hecke characters of $\mathbb{Q}$ of finite order correspond precisely to primitive Dirichlet characters of $\mathbb{Q}$.
- In fact, any Hecke character of $\mathbb{Q}$ is of the form $\eta|\cdot|_{\mathbb{A}_K}^s$ for some $s \in \mathbb{C}$, where η is a Hecke character of finite order.
- In general, a Hecke character $\chi : C_K \to \mathbb{C}^\times$ is uniquely determined by local multiplicative characters $\chi|_{K_v^\times} : K_v^\times \to \mathbb{C}^\times$, which are unramified, so $\chi|_{K_v^\times}(\mathcal{O}_v^\times) = 1$, for almost all $v \in V_K$.

Global theory — Hecke characters

A **Hecke character** is a character of the idèle class group, that is a continuous homomorphism $C_K \to \mathbb{C}^\times$ with the discrete topology on $\mathbb{C}^\times$.

A **Hecke L-function** of χ is

$$L(\chi) := \prod_{v \in V_K^f} L_{K_v}(\chi|_{K_v^\times}),$$

where L_{K_v} are the local L-factors

$$L_{K_v}(\chi) = \begin{cases} (1 - \chi(\pi_v))^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is not unramified.} \end{cases}$$

Examples

- If $\chi = |\cdot|_{\mathbb{A}_K}^s$, then $L(\chi)$ is the Dedekind ζ-function $\zeta_K(s)$.
- If $K = \mathbb{Q}$ and χ has finite order, then $L(\chi)$ is the Dirichlet L-function of a primitive Dirichlet character.

Global theory — Fourier analysis

The three components for the global Fourier transform are simply defined as the product of their local counterparts with the unramified condition.

- The global Schwartz–Bruhat functions on $\mathbb{A}_K$ are linear combinations of products of local Schwartz–Bruhat functions $f_v : K_v \to \mathbb{C}$ such that $f_v = \mathbb{I}_{\mathcal{O}_v}$ for almost all $v \in V_K$.
- The global Haar measure on $\mathbb{A}_K$ is such that

$$\int_{\mathbb{A}_K} f(x)\mathrm{d}_{\mathbb{A}_K} x := \prod_{v \in V_K} \int_{K_v} f|_{K_v}(x)\mathrm{d}_{K_v} x.$$

- The global additive character on $\mathbb{A}_K$ is such that

$$\psi_{\mathbb{A}_K}((x_v)_{v \in V_K}) := \prod_{v \in V_K} \psi_{K_v}(x_v).$$

By construction, since the Fourier inversion formula holds in all completions of K, the Poisson summation formula holds in $\mathbb{A}_K$.

Global theory — ζ-integrals

Let $f : \mathbb{A}_K \to \mathbb{C}$ be a Schwartz–Bruhat function, and let $\chi : C_K \to \mathbb{C}^\times$ be a Hecke character. The **global ζ-integral** is defined to be

$$\zeta(f, \chi) := \prod_{v \in V_K} \zeta_{K_v}(f|_{K_v^\times}, \chi|_{K_v^\times}),$$

which is an infinite product.

Theorem (Functional equation for the global ζ-integral)

ζ has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation

$$\zeta(f, \chi) = \zeta(\widehat{f}, \chi^{-1}|\cdot|_{\mathbb{A}_K}).$$

Sketch of proof.

The Poisson summation formula $\mathbb{A}_K$ relates f and $\widehat{f}$. $\square$

Global theory — ζ-integrals

Theorem (Tate (1950))

$L(\chi)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation $\Lambda(\chi) = \epsilon(\chi)\Lambda(\chi^{-1}|\cdot|_{\mathbb{A}_K})$ where

$$\Lambda(\chi) := L_{\mathbb{R}}(s)^{r_1} \cdot L_{\mathbb{C}}(s)^{r_2} \cdot L(\chi), \qquad \epsilon(\chi) := \prod_{v \in V_K} \epsilon_{K_v}(\chi).$$

Here $\epsilon(\chi)$ is the **global ϵ-factor**, and similarly the **global root number** is defined to be $w(\chi) := \prod_{v \in V_K} w_{K_v}(\chi) \in U(1)$.

Proof.

The product of the functional equations for the local ζ-integrals is

$$\frac{\zeta(\widehat{f}, \chi^{-1}|\cdot|_{\mathbb{A}_K})}{\Lambda(\chi^{-1}|\cdot|_{\mathbb{A}_K})} = \epsilon(\chi)\frac{\zeta(f, \chi)}{\Lambda(\chi)}.$$

Divide this by the functional equation for the global ζ-integral. □