Galois representations and root numbers

Tuesday, 22 November 2022

Tate's thesis [1] and epsilon factors

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[1] Tate (1950) *Fourier analysis in number fields and Hecke's zeta-functions*

Overview

Consider the Riemann ζ-function

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Sketch of proof.

Write $Z(s)$ as a real integral of the theta series $\Theta(z) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 z}$.
The Poisson summation formula for $\mathbb{Z} \subset \mathbb{R}$ relates $\Theta(z)$ and $\Theta(1/z)$. $\square$

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Write $Z_K(s)$ as a real integral of a generalised theta series $\Theta_K(s)$ and apply the Poisson summation formula for a lattice in $\mathbb{R}^n$. □

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Note that there is an Euler product

$$\zeta_K(s) = \prod_{v \in V_K^f} \frac{1}{1 - q_v^{-s}} = \prod_{v \in V_K^f} \left(\sum_{n=0}^{\infty} q_v^{-ns} \right),$$

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Idea: the global ζ-integral over $\mathbb{A}_K^\times$ is the product of local ζ-integrals over $K_v^\times$, and the Γ-factors are local ζ-integrals at the archimedean places.

Local theory — Fourier analysis

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Each of these can be generalised for $F = \mathbb{C}$ and $F/\mathbb{Q}_p$.

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  so that $\mu_{\mathbb{Q}_p^\times}(\mathbb{Z}_p^\times) = 1$. If $G/\mathbb{Q}_p$, then μ_G and $\mu_{G^\times}$ should account for the valuation δ_v of the different ideal $\mathcal{D}_{G/\mathbb{Q}_p} \trianglelefteq \mathcal{O}_G$.

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$$\begin{aligned} \int_{\mathbb{R}^\times} f(x)|x|_{\mathbb{R}}^s \, \mathrm{d}_{\mathbb{R}^\times} x &= 2 \int_0^\infty e^{-\pi x^2} x^s \, \frac{\mathrm{d}x}{x} \\ &= \int_0^\infty e^{-y} \left(\frac{y}{\pi}\right)^{\frac{s}{2}} \frac{\mathrm{d}y}{y} \qquad\qquad y = \pi x^2 \\ &= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) =: \Gamma_{\mathbb{R}}(s). \end{aligned}$$

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$$\int_{\mathbb{C}^\times} f(z) |z|_{\mathbb{C}}^s \, \mathrm{d}_{\mathbb{C}^\times} z = \cdots = 2(2\pi)^{-s}\Gamma(s) =: \Gamma_{\mathbb{C}}(s).$$

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- If $F = K_v/\mathbb{Q}_p$, this is a linear combination of characteristic functions

$$\mathbb{I}_{a+\pi_v^n\mathcal{O}_v}(x) = \begin{cases} 1 & x \in a + \pi_v^n\mathcal{O}_v \\ 0 & x \notin a + \pi_v^n\mathcal{O}_v \end{cases}.$$

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$$= \sum_{n=0}^{\infty} \frac{p^{n-ns}}{1-p^{-1}} \int_{p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p} \mathrm{d}_{\mathbb{Q}_p} x = \sum_{n=0}^{\infty} p^{-ns}.$$

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If $F/\mathbb{Q}_p$, let $f_0(x) := \mathbb{I}_{\mathcal{O}_F}(x)$ instead.

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A Schwartz-Bruhat function $f : F \to \mathbb{C}$ has a Fourier transform

$$\widehat{f}(y) := \int_F \psi_F(xy) f(x) \, \mathrm{d}_F x,$$

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Local theory — additive characters

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These are defined in such a way so that the Fourier inversion formula $\widehat{\widehat{f}}(x) = f(-x)$ holds, giving a duality between ψ_F and $\mathrm{d}_F x$. Indeed $\widehat{\widehat{f_0}} = f_0$, which is necessary in the Poisson summation formula.

Local theory — ζ-integrals

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$\zeta_F(f_0, |\cdot|_F^s)$ are the Γ-factors and local Euler factors.

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$\zeta_F(f_0, |\cdot|_F^s)$ are the Γ-factors and local Euler factors.

Theorem (Functional equation for the local ζ-integral)

There is a meromorphic function $L_F : \mathrm{Hom}_{\mathrm{cts}}(F^\times, \mathbb{C}^\times) \to \mathbb{C}^\times$ and a holomorphic function $\epsilon_F : \mathrm{Hom}_{\mathrm{cts}}(F^\times, \mathbb{C}^\times) \to \mathbb{C}^\times$ such that

$$\frac{\zeta_F(\widehat{f}, \chi^{-1}|\cdot|_F)}{L_F(\chi^{-1}|\cdot|_F)} = \epsilon_F(\chi)\frac{\zeta_F(f, \chi)}{L_F(\chi)}.$$

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The **local root number** is then defined to be

$$w_F(\chi) := \frac{\epsilon_F(\chi)}{|\epsilon_F(\chi)|} \in U(1).$$

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  - If $\eta = \mathrm{sgn}$, set $f(x) := xe^{-\pi x^2}$ and $L_{\mathbb{R}}(\chi) := \Gamma_{\mathbb{R}}(s+1)$. Then compute $\epsilon_{\mathbb{R}}(\chi) = -i$.

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- Let $F = \mathbb{C}$. Then

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- Let $F = \mathbb{C}$. Then

$$\chi(z) = (z/\sqrt{z\bar{z}})^n |z|_{\mathbb{C}}^s, \qquad n \in \mathbb{Z}.$$

  - If $n = 0$, set $f(z) := f_0(z) = \frac{1}{\pi} e^{-2\pi z\bar{z}}$ and $L_{\mathbb{C}}(\chi) := \Gamma_{\mathbb{C}}(s)$. Then compute $\epsilon_{\mathbb{C}}(\chi) = 1$.

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  - In general, set $f(z) := \frac{1}{\pi} z^n e^{-2\pi z\bar{z}}$ and $L_{\mathbb{C}}(\chi) := \Gamma_{\mathbb{C}}(s + \frac{1}{2}|n|)$. Then compute $\epsilon_{\mathbb{C}}(\chi) = i^{-|n|}$.

Local theory — ϵ-factors

How do you compute $\epsilon_F(\chi)$?

Determine multiplicative characters $\chi : F^\times \to \mathbb{C}^\times$ completely.

- Let $F = K_v/\mathbb{Q}_p$. The **conductor** of χ is the least $n \in \mathbb{N}$ such that

$$\chi((1 + \pi_v^n \mathcal{O}_v) \cap \mathcal{O}_v^\times) = 1.$$

  If $n = 0$, then χ is said to be **unramified**.

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  - If $n = 0$, set $f := \mathbb{I}_{\mathcal{O}_v}$ and $L_{K_v}(\chi) := (1 - \chi(\pi_v)^{-1})^{-1}$. Then compute

$$\epsilon_{K_v}(\chi) = q_v^{\frac{\delta_v}{2}} \chi(\pi_v)^{\delta_v}.$$

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  - If $n > 0$, set $f := \mathbb{I}_{1+\pi_v^n \mathcal{O}_v}$ and $L_{K_v}(\chi) := 1$.
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F	χ	$L_F(\chi)$	$\epsilon_F(\chi)$
$\mathbb{R}$	$\lvert x\rvert_{\mathbb{R}}^s$	$\Gamma_{\mathbb{R}}(s)$	1
$\mathbb{R}$	$\mathrm{sgn}(x)\lvert x\rvert_{\mathbb{R}}^s$	$\Gamma_{\mathbb{R}}(s+1)$	$-i$
$\mathbb{C}$	$(z/\sqrt{z\bar{z}})^n\lvert z\rvert_{\mathbb{C}}^s$	$\Gamma_{\mathbb{C}}(s+\frac{1}{2}\lvert n\rvert)$	$i^{-\lvert n\rvert}$
K_v	unramified	$(1-\chi(\pi_v)^{-1})^{-1}$	$q_v^{\frac{\delta_v}{2}}\chi(\pi_v)^{\delta_v}$
K_v	ramified	1	$\int_{K_v^\times}\psi_v(x)\chi(x)^{-1}\,\mathrm{d}_{K_v}x$

Global theory — adèles and idèles

Let $V_K = V_K^f \cup V_K^\infty$ be the set of places of a number field K.

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Consider the adèle ring

$$\mathbb{A}_K := \left\{ (x_v)_{v \in V_K} \in \prod_{v \in V_K} K_v \;\middle|\; x_v \in \mathcal{O}_v \text{ for almost all } v \in V_K \right\}.$$

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Its unit group is the idèle group

$$\mathbb{A}_K^\times := \left\{ (x_v)_{v \in V_K} \in \prod_{v \in V_K} K_v^\times \;\middle|\; x_v \in \mathcal{O}_v^\times \text{ for almost all } v \in V_K \right\}.$$

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Example

If $K = \mathbb{Q}$, then

$$\mathbb{A}_\mathbb{Q} \cong \mathbb{R} \times \bigcup_{n \in \mathbb{N}^+} \frac{1}{n} \prod_{p < \infty} \mathbb{Z}_p.$$

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Let $V_K = V_K^f \cup V_K^\infty$ be the set of places of a number field K.

The idèle group is endowed with the restricted product topology such that

$$\prod_{v \in S} U_v \times \prod_{v \in V_K \setminus S} \mathcal{O}_v^\times,$$

is an open basis for some finite $V_K^\infty \subseteq S \subset V_K$ and some open $U_v \subseteq K_v^\times$.

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There is a diagonal embedding $K^\times \hookrightarrow \mathbb{A}_K^\times$. By the product formula,

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By Tychonoff's theorem, both the idèle group $\mathbb{A}_K^\times$ and the idèle class group $C_K := \mathbb{A}_K^\times / K^\times$ are locally compact topological groups.

Global theory — Hecke characters

A **Hecke character** is a character of the idèle class group, that is a continuous homomorphism $C_K \to \mathbb{C}^\times$ with the discrete topology on $\mathbb{C}^\times$.

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Examples

- A Dirichlet character $\phi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}^\times$ induces a Hecke character

$$C_{\mathbb{Q}} \cong \mathbb{R}^+ \times \prod_{p<\infty} \mathbb{Z}_p^\times \twoheadrightarrow \prod_{p|n} (\mathbb{Z}_p/n\mathbb{Z}_p)^\times \cong (\mathbb{Z}/n\mathbb{Z})^\times \xrightarrow{\phi} \mathbb{C}^\times$$

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- In fact, any Hecke character of $\mathbb{Q}$ is of the form $\eta|\cdot|_{\mathbb{A}_K}^s$ for some $s \in \mathbb{C}$, where η is a Hecke character of finite order.
- In general, a Hecke character $\chi : C_K \to \mathbb{C}^\times$ is uniquely determined by local multiplicative characters $\chi|_{K_v^\times} : K_v^\times \to \mathbb{C}^\times$, which are unramified, so $\chi|_{K_v^\times}(\mathcal{O}_v^\times) = 1$, for almost all $v \in V_K$.

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A **Hecke character** is a character of the idèle class group, that is a continuous homomorphism $C_K \to \mathbb{C}^\times$ with the discrete topology on $\mathbb{C}^\times$.

A **Hecke L-function** of χ is

$$L(\chi) := \prod_{v \in V_K^f} L_{K_v}(\chi|_{K_v^\times}),$$

where L_{K_v} are the local L-factors

$$L_{K_v}(\chi) = \begin{cases} (1-\chi(\pi_v))^{-1} & \chi \text{ is unramified} \\ 1 & \chi \text{ is not unramified} \end{cases}.$$

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$$L_{K_v}(\chi) = \begin{cases} (1-\chi(\pi_v))^{-1} & \chi \text{ is unramified} \\ 1 & \chi \text{ is not unramified} \end{cases}.$$

Examples

- If $\chi = |\cdot|_{\mathbb{A}_K}^s$, then $L(\chi)$ is the Dedekind ζ-function $\zeta_K(s)$.

Global theory — Hecke characters

A **Hecke character** is a character of the idèle class group, that is a continuous homomorphism $C_K \to \mathbb{C}^\times$ with the discrete topology on $\mathbb{C}^\times$.

A **Hecke L-function** of χ is

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Examples

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- If $K = \mathbb{Q}$ and χ has finite order, then $L(\chi)$ is the Dirichlet L-function of a primitive Dirichlet character.

Global theory — Fourier analysis

The three components for the global Fourier transform are simply defined as the product of their local counterparts with the unramified condition.

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- The global Schwartz-Bruhat functions on $\mathbb{A}_K$ are linear combinations of products of local Schwartz-Bruhat functions $f_v : K_v \to \mathbb{C}$ for all $v \in V_K$, such that $f_v = \mathbb{I}_{\mathcal{O}_v}$ for almost all $v \in V_K$.

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- The global Haar measure on $\mathbb{A}_K$ is such that

$$\int_{\mathbb{A}_K} f(x) \, \mathrm{d}_{\mathbb{A}_K} x := \prod_{v \in V_K} \int_{K_v} f|_{K_v}(x) \, \mathrm{d}_{K_v} x.$$

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By construction, since the Fourier inversion formula holds in all completions of K, the Poisson summation formula holds in $\mathbb{A}_K$.

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Sketch of proof.

The Poisson summation formula $\mathbb{A}_K$ relates f and $\widehat{f}$. □

Global theory — ζ-integrals

Theorem (Tate (1950))

$L(\chi)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation $\Lambda(\chi) = \epsilon(\chi)\Lambda(\chi^{-1}|\cdot|_{\mathbb{A}_K})$ where

$$\Lambda(\chi) := L_{\mathbb{R}}(s)^{r_1} \cdot L_{\mathbb{C}}(s)^{r_2} \cdot L(\chi), \qquad \epsilon(\chi) := \prod_{v \in V_K} \epsilon_{K_v}(\chi).$$

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Proof.

The product of the functional equations for the local ζ-integrals is

$$\frac{\zeta(\widehat{f}, \chi^{-1}|\cdot|_{\mathbb{A}_K})}{\Lambda(\chi^{-1}|\cdot|_{\mathbb{A}_K})} = \epsilon(\chi)\frac{\zeta(f, \chi)}{\Lambda(\chi)}.$$

Divide this by the functional equation for the global ζ-integral. □

Thank you!