The Tate–Shafarevich and Brauer groups

Curves over function fields

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Overview

Part I

- The Tate–Shafarevich group of a $\begin{cases}\text{number field}\\ \text{function field}\end{cases}$
- The Artin–Tate conjecture

Part II

- The Brauer–$\begin{cases}\text{Grothendieck}\\ \text{Azumaya}\end{cases}$ group of a $\begin{cases}\text{field}\\ \text{scheme}\end{cases}$
- The Brauer–Manin obstruction

The Tate–Shafarevich group of a number field

Let E be an elliptic curve over a number field K. Let

$$V_K := \{\text{closed points of } \operatorname{Spec}(\mathcal{O}_K)\} \cup V_K^{\infty}.$$

The **Tate–Shafarevich group** is

$$Ш(E/K) := \ker\left(H^1(K,E) \to \prod_{v \in V_K} H^1(K_v, E)\right).$$

Note that there is a bijection

$$H^1(K,E) \xrightarrow{\sim} \mathrm{WC}(E/K),$$

the **Weil–Châtelet group** of torsors for E/K. Thus $0 \neq C \in Ш(E/K)$ is a K-twist of E that is everywhere locally soluble but globally insoluble.

Example (Selmer)

The curve $3X^3 + 4Y^3 + 5Z^3 = 0$ is a $\mathbb{Q}$-twist of $E : X^3 + Y^3 + 60Z^3 = 0$ that is everywhere locally soluble but globally insoluble, so $Ш(E/\mathbb{Q}) \neq 0$.

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Conjecture (Tate–Shafarevich)

$\#Ш(E/K)$ *is finite.*

Conjecture (Birch–Swinnerton-Dyer)

Assuming TS holds,

$$\lim_{s \to 1} \frac{L(E/K, s)}{(s-1)^{\operatorname{rk}(E/K)}} = \frac{R \cdot \#Ш(E/K) \cdot \tau}{\#E(K)_{\text{tor}}^2}.$$

The Tate–Shafarevich group of a function field

Let E be an elliptic curve over a function field $K = \mathbb{F}_q(C)$. Let

$$V_K := \{\text{closed points of } C\}.$$

The **Tate–Shafarevich group** is

$$Ш(E/K) := \ker\left(H^1(K,E) \to \prod_{v \in V_K} H^1(K_v, E)\right).$$

Conjecture (Tate–Shafarevich)

$\#Ш(E/K)$ *is finite.*

Theorem (KT03)

Assuming $TS[\ell^\infty]$ *holds for some* ℓ*,*

$$\lim_{s\to 1} \frac{L(E/K,s)}{(s-1)^{\mathrm{rk}(E/K)}} = \frac{R \cdot \#Ш(E/K) \cdot \tau}{\#E(K)^2_{\mathrm{tor}}}.$$

The Tate–Shafarevich group of a function field

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Theorem (Mil68, Mil70, ASD73)

TS holds if $\mathcal{E}$ is constant, rational, or K3.

Theorem (Ulm12, Proposition 5.3.1)

$\mathrm{Br}(\mathcal{E}) \xrightarrow{\sim} Ш(E/K)$.

The Artin–Tate conjecture

Let $\mathcal{E} \to C$ be an elliptic surface over $\mathbb{F}_q$ with generic fibre E/K. Then

$$\begin{aligned} \text{BSD holds for } E \quad &\overset{\text{KT03}}{\Longleftrightarrow} \quad \#Ш(E/K)[\ell^\infty] \text{ is finite for some } \ell \\ &\overset{\text{Gro79}}{\Longleftrightarrow} \quad \#\operatorname{Br}(\mathcal{E})[\ell^\infty] \text{ is finite for some } \ell \\ &\overset{\text{Mil75}}{\Longleftrightarrow} \quad \text{AT (and T) holds for } \mathcal{E}. \end{aligned}$$

Conjecture (Artin–Tate)

Let X be a smooth projective geometrically-connected surface over $\mathbb{F}_q$. Then $\#\operatorname{Br}(X)$ is finite, and if $\operatorname{NS}(X)_{/\operatorname{tor}} = \langle D_i \rangle$, then

$$\lim_{s\to 1} \frac{P_2(X, q^{-s})}{(1-q^{1-s})^{\operatorname{rk}(\operatorname{NS}(X))}} = \frac{\#\operatorname{Br}(X) \cdot |\det(\langle D_i, D_j\rangle_{i,j})|}{\#\operatorname{NS}(X)^2_{\operatorname{tor}} \cdot q^{\chi(X,\mathcal{O}_X)-1+\dim \operatorname{Pic}(X)}}.$$

Note that if $X \to C$ is flat proper with smooth geometrically-connected generic fibre X_K/K, then $\#Ш(\operatorname{Jac}(X_K)/K) \sim \#\operatorname{Br}(X)$ (LLR18).

The Brauer–Azumaya group of a field

Let K be a field. The **classical Brauer group** of K is

$$\mathrm{Br}(K) := \{\text{central simple algebras over } K\}/\sim .$$

A **central simple algebra** over K is a finite-dimensional associative K-algebra with centre K and no non-trivial proper two-sided ideals.

Examples

- Algebra of $n \times n$ matrices $\mathrm{Mat}_n(K)$ over K.
- Algebra of $n \times n$ matrices $\mathrm{Mat}_n(D)$ over a central division algebra D.
- Tensor product $A \otimes_K B$ of two CSAs A and B.
- Opposite algebra A^{op} of a CSA A.

Two CSAs A and B over K are **equivalent** if there are $n, m \in \mathbb{N}$ such that $A \otimes_K \mathrm{Mat}_n(K) \cong B \otimes_K \mathrm{Mat}_m(K)$.

Example

If $n, m \in \mathbb{N}$ and D is a CDA, then $\mathrm{Mat}_n(D) \sim \mathrm{Mat}_m(D)$.

The Brauer–Azumaya group of a field

Let K be a field. The **classical Brauer group** of K is

$$\mathrm{Br}(K) := \{\text{central simple algebras over } K\}/\sim .$$

Examples

- $\mathrm{Br}(\mathbb{F}_q) = 0$. Suffices to prove a CDA D over $\mathbb{F}_q$ is $\mathbb{F}_q$. A finite division algebra D is a field K. A field K with centre $\mathbb{F}_q$ is $\mathbb{F}_q$.
- $\mathrm{Br}(\mathbb{C}) = 0$. Suffices to prove a CDA D over $\mathbb{C}$ is $\mathbb{C}$. If $x \in D$, then $\mathbb{C}[x]$ is an integral domain and a finite-dimensional $\mathbb{C}$-vector space. Thus $\mathbb{C}[x]$ is a field, but $\mathbb{C}$ does not have finite extensions.
- $\mathrm{Br}(\mathbb{C}(X)) = 0$ for a curve $X/\mathbb{C}$. This is Tsen's theorem.

The Brauer–Grothendieck group of a field

Let K be a field. The **cohomological Brauer group** of K is

$$\mathrm{Br}'(K) := H^2(K, \mathbb{G}_m).$$

Theorem (CTS19, Theorem 1.3.5)

$\mathrm{Br}(K) \xrightarrow{\sim} \mathrm{Br}'(K)$.

Examples

- $\mathrm{Br}'(\mathbb{R}) = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. By cohomology of cyclic groups,

  $$\mathrm{Br}'(\mathbb{R}) = H^2(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) \cong \mathbb{R}^\times / \mathrm{Nm}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) \cong \{\pm\}.$$

  In fact, $\mathrm{Br}'(\mathbb{R}) = \{\mathbb{R}, \mathbb{H}\}$.
- Local class field theory gives isomorphisms

  $$\mathrm{inv}_p : \mathrm{Br}'(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}, \qquad \mathrm{inv}_q : \mathrm{Br}'(\mathbb{F}_q((T))) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

The Brauer–Grothendieck group of a field

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Examples

- Global class field theory gives short exact sequences

$$0 = \varinjlim_{L/K} H^1(L/K, C_L) \to \mathrm{Br}'(\mathbb{Q}) \to \bigoplus_{v \in V_{\mathbb{Q}}} \mathrm{Br}'(\mathbb{Q}_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0,$$

$$0 = H^1(\mathbb{F}_q, \mathrm{Jac}(C_{\overline{\mathbb{F}_q}})) \to \mathrm{Br}'(K) \to \bigoplus_{v \in V_K} \mathrm{Br}'(K_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0,$$

  where $K = \mathbb{F}_q(C)$.

The Brauer–Azumaya group of a scheme

Let X be a scheme. The **Brauer–Azumaya group** of X is

$$\mathrm{Br}_{\mathrm{Az}}(X) := \{\text{Azumaya algebras on } X\}/\sim .$$

An **Azumaya algebra** $\mathcal{A}$ **on** X is a locally free $\mathcal{O}_X$-algebra of finite type such that $\mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \kappa_x$ is a CSA over κ_x for all closed points $x \in X$.

Examples

- Trivial, tensor product, opposite algebra sheaves of AAs.
- $(X = \mathrm{Spec}(K))$ For a CSA A over K, the constant sheaf A.
- $(X = \mathbb{P}^n_K)$ For a CSA A over K, the sheaf $A \otimes_K \mathcal{E}nd_K(\bigoplus_{n_i} \mathcal{O}_X(n_i))$.

Two AAs $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** if there are locally free $\mathcal{O}_X$-modules A and B of finite rank such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(A) \cong \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(B)$.

Examples

- $\mathrm{Br}_{\mathrm{Az}}(\mathrm{Spec}(K)) = \mathrm{Br}(K)$.
- (Fis17) $\mathrm{Br}_{\mathrm{Az}}(C)$ for an smooth curve of genus one C/K.

The Brauer–Grothendieck group of a scheme

Let X be a scheme. The **Brauer–Grothendieck group** of X is

$$\mathrm{Br}_{\mathrm{Gr}}(X) := H^2_{\text{ét}}(X, \mathbb{G}_m).$$

Unlike for fields, in general $\mathrm{Br}_{\mathrm{Az}}(X) \hookrightarrow \mathrm{Br}_{\mathrm{Gr}}(X)$ is not surjective.

Theorem (CTS19, Theorem 3.3.2)

Assume X is quasi-compact separated with an ample line bundle. Then

$$\mathrm{Br}(X) := \mathrm{Br}_{\mathrm{Az}}(X) \xrightarrow{\sim} \mathrm{Br}_{\mathrm{Gr}}(X)_{\mathrm{tor}}.$$

Example

A quasi-projective scheme over an affine scheme, such as $E/\mathbb{F}_q(C)$ or $\mathcal{E}/\mathbb{F}_q$. If X is regular integral noetherian, then $\mathrm{Br}_{\mathrm{Gr}}(X)$ is already torsion.

Theorem (CTS19, Theorem 3.5.4)

Assume X is regular integral over a field K. Then $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(K(X))$.

The Brauer–Grothendieck group of a scheme

Let X be a variety over a perfect field K, and write $\overline{X} := X \times_K \overline{K}$. The first seven terms of the Leray spectral sequence form an exact sequence

$$0 \longrightarrow H^1(K, \overline{K}[X]^\times) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\overline{X})^{G_K} \longrightarrow H^2(K, \overline{K}[X]^\times) \longrightarrow$$

$$\longrightarrow \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})) \to H^1(K, \mathrm{Pic}(\overline{X})) \to \ker(H^3(K, \overline{K}[X]^\times) \to H^3_{\text{ét}}(X, \mathbb{G}_m)).$$

Examples

- If $X = \mathbb{A}^1_K$ or $X = \mathbb{P}^1_K$, then $\mathrm{Br}(X) \cong \mathrm{Br}(K)$.
  - $H^2(K, \overline{K}[X]^\times) \cong \mathrm{Br}(K)$ since $\overline{K}[X]^\times = \overline{K}^\times$.
  - $\mathrm{Br}(\overline{X}) \hookrightarrow \mathrm{Br}(\overline{K}(X)) = 0$ by Tsen's theorem.
  - $\mathrm{Br}(K) \to \mathrm{Br}(X)$ and $H^3(K, \overline{K}[X]^\times) \to H^3_{\text{ét}}(X, \mathbb{G}_m)$ are injective since $X(K) \neq \emptyset$ gives retractions.
  - $H^1(K, \mathrm{Pic}(\overline{X})) = 0$ since $\mathrm{Pic}(\mathbb{A}^1_{\overline{K}}) = 0$ and $\deg : \mathrm{Pic}(\mathbb{P}^1_{\overline{K}}) \xrightarrow{\sim} \mathbb{Z}$.

  In fact, $\mathrm{Br}(\mathbb{A}^n_K) \cong \mathrm{Br}(\mathbb{P}^n_K) \cong \mathrm{Br}(K)$ by induction.

The Brauer–Grothendieck group of a scheme

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$$\longrightarrow \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})) \to H^1(K, \mathrm{Pic}(\overline{X})) \to \ker(H^3(K, \overline{K}[X]^\times) \to H^3_{\text{ét}}(X, \mathbb{G}_m)).$$

Examples

- If $X = E$ is an elliptic curve, then there is a short exact sequence

  $$0 \to \mathrm{Br}(K) \to \mathrm{Br}(E) \to H^1(K, E) \to 0.$$

  As before, with $H^1(K, \mathrm{Pic}(\overline{E})) = H^1(K, \mathrm{Jac}(\overline{E})) = H^1(K, E)$.
- (Tho10) $\mathrm{Br}(\mathcal{E})[\ell^\infty]$ for an elliptic K3 surface $\mathcal{E}/\mathbb{F}_q$ given by $t(t-1)y^2 = x(x-1)(x-t)$. Uses the short exact sequence

  $$0 \to \mathrm{NS}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^2_{\text{ét}}(\mathcal{E}, \mathbb{Z}_\ell(1)) \to T_\ell \, \mathrm{Br}(\mathcal{E}) \to 0.$$

The Brauer–Manin obstruction

Let X be a scheme over a global field K. A point $x_v : \mathrm{Spec}(K_v) \to X$ induces a map $x_v^* : \mathrm{Br}(X) \to \mathrm{Br}(K_v)$. The **Brauer–Manin pairing** is

$$\begin{array}{rcrcl} \langle -, - \rangle_{\mathrm{Br}} & : & \mathrm{Br}(X) \times X(\mathbb{A}_K) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ & & (A, (x_v)_v) & \longmapsto & \displaystyle\sum_{v \in V_K} \mathrm{inv}_v(x_v^*(A)) \end{array} .$$

The **Brauer–Manin set** for $A \in \mathrm{Br}(X)$ is

$$X(\mathbb{A}_K)^A := \{(x_v) \in X(\mathbb{A}_K) : \langle A, (x_v)_v \rangle_{\mathrm{Br}} = 0\}.$$

By global class field theory,

$$\overline{X(K)} \hookrightarrow \bigcap_{A \in \mathrm{Br}(X)} X(\mathbb{A}_K)^A \subseteq X(\mathbb{A}_K).$$

If $X(\mathbb{A}_K)^A \neq \emptyset$ but $X(\mathbb{A}_K) = \emptyset$, then there is a **Brauer–Manin obstruction to the Hasse principle** for X due to $A \in \mathrm{Br}(X)$.

The Brauer–Manin obstruction

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$$\begin{array}{rcrcl} \langle -,-\rangle_{\mathrm{Br}} & : & \mathrm{Br}(X) \times X(\mathbb{A}_K) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ & & (A,(x_v)_v) & \longmapsto & \displaystyle\sum_{v \in V_K} \mathrm{inv}_v(x_v^*(A)) \end{array} .$$

Theorem (Wit15)

Let $\mathcal{E}$ be an elliptic K3 surface over $\mathbb{Q}$ given by

$$y^2 = x(x-3(t-1)^3(3+t))(x+3(t+1)^3(3-t)).$$

There is a Brauer–Manin obstruction to the Hasse principle for $\mathcal{E}$ due to

$$(x+3(t-1)^3(3+t), 6t(t+1)) + (x-3(t+1)^3(3-t), 6t(t-1)) \in \mathrm{Br}(\mathcal{E}).$$

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