University College London

Curves over function fields

The Tate-Shafarevich and Brauer groups

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Overview

Part I

- The Tate-Shafarevich group of a $\begin{cases} \text{number field} \\ \text{function field} \end{cases}$
- The Artin-Tate conjecture

Part II

- The Brauer-$\begin{cases} \text{Grothendieck} \\ \text{Azumaya} \end{cases}$ group of a $\begin{cases} \text{field} \\ \text{scheme} \end{cases}$
- The Brauer-Manin obstruction

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the **Weil-Châtelet group** of torsors for E/K.

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Example (Selmer)

The curve $3X^3 + 4Y^3 + 5Z^3 = 0$ is a $\mathbb{Q}$-twist of $E : X^3 + Y^3 + 60Z^3 = 0$ that is everywhere locally soluble but globally insoluble, so $Ш(E/\mathbb{Q}) \neq 0$.

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Conjecture (Birch-Swinnerton-Dyer)

Assuming TS holds,

$$\lim_{s \to 1} \frac{L(E/K, s)}{(s-1)^{\mathrm{rk}(E/K)}} = \frac{R \cdot \#\text{Ш}(E/K) \cdot \tau}{\#E(K)_{\mathrm{tors}}^2}.$$

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Let E be an elliptic curve over a function field $K = \mathbb{F}_q(C)$. Let

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Theorem (KT03)

Assuming $TS[\ell^\infty]$ *holds for some* ℓ,

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TS holds if and only if $TS[\ell^\infty]$ *holds for some* ℓ.

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Theorem (Ulm12, Proposition 5.3.1)

$\mathrm{Br}(\mathcal{E}) \xrightarrow{\sim} \text{Ш}(E/K)$.

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Let X be a smooth projective geometrically-connected surface over $\mathbb{F}_q$. Then $\#\mathrm{Br}(X)$ is finite, and if $\mathrm{NS}(X)_{/\mathrm{tors}} = \langle D_i \rangle$, then

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Note that if $X \to C$ is flat proper with smooth geometrically-connected generic fibre X_K/K, then $\#\text{Ш}(\mathrm{Jac}(X_K)/K) \sim \#\mathrm{Br}(X)$ (LLR18).

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Two CSAs A and B over K are **equivalent** if there are $n, m \in \mathbb{N}$ such that $A \otimes_K \mathrm{Mat}_n(K) \cong B \otimes_K \mathrm{Mat}_m(K)$.

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- $\mathrm{Br}(\mathbb{C}(X)) = 0$ for a curve $X/\mathbb{C}$. This is Tsen's theorem.

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- Local class field theory gives isomorphisms

$$\mathrm{inv}_p : \mathrm{Br}'(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}, \qquad \mathrm{inv}_q : \mathrm{Br}'(\mathbb{F}_q((T))) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.$$

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Theorem (CTS19, 4.8)

The first seven terms form an exact sequence

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- (Tho10) $\mathrm{Br}(\mathcal{E})[\ell^\infty]$ for an elliptic K3 surface $\mathcal{E}/\mathbb{F}_q$ given by $t(t-1)y^2 = x(x-1)(x-t)$.

The Brauer-Grothendieck group of a scheme

Let X be a variety over a perfect field K. There is an exact sequence

$$0 \longrightarrow H^1(K, \overline{K}[X]^\times) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\overline{X})^{G_K} \longrightarrow H^2(K, \overline{K}[X]^\times) \longrightarrow \ker(\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})) \to H^1(K, \mathrm{Pic}(\overline{X})) \to \ker(H^3(K, \overline{K}[X]^\times) \to H^3_{\text{ét}}(X, \mathbb{G}_m)).$$

Examples

- If $X = E$ is an elliptic curve, then there is a short exact sequence

  $$0 \to \mathrm{Br}(K) \to \mathrm{Br}(E) \to H^1(K, E) \to 0.$$

  As before, with $H^1(K, \mathrm{Pic}(\overline{E})) = H^1(K, \mathrm{Jac}(\overline{E})) = H^1(K, E)$.
- (Tho10) $\mathrm{Br}(\mathcal{E})[\ell^\infty]$ for an elliptic K3 surface $\mathcal{E}/\mathbb{F}_q$ given by $t(t-1)y^2 = x(x-1)(x-t)$. Uses the short exact sequence

  $$0 \to \mathrm{NS}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to H^2_{\text{ét}}(\mathcal{E}, \mathbb{Z}_\ell(1)) \to T_\ell \mathrm{Br}(\mathcal{E}) \to 0,$$

  obtained by applying ℓ-adic cohomology to the Kummer sequence.

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$$\begin{array}{rccl} \langle -, - \rangle_{\mathrm{Br}} & : & \mathrm{Br}(X) \times X(\mathbb{A}_K) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ & & (A, (x_v)_v) & \longmapsto & \sum_{v \in V_K} \mathrm{inv}_v(x_v^*(A)) \end{array} .$$

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If $X(\mathbb{A}_K)^A \neq \emptyset$ but $X(\mathbb{A}_K) = \emptyset$, then there is a **Brauer-Manin obstruction to the Hasse principle** for X due to $A \in \mathrm{Br}(X)$.

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Theorem (Wit15)

Let $\mathcal{E}$ be an elliptic K3 surface over $\mathbb{Q}$ given by

$$y^2 = x(x - 3(t-1)^3(3+t))(x + 3(t+1)^3(3-t)).$$

There is a Brauer-Manin obstruction to the Hasse principle for $\mathcal{E}$ due to

$$(x + 3(t-1)^3(3+t), 6t(t+1)) + (x - 3(t+1)^3(3-t), 6t(t-1)) \in \mathrm{Br}(\mathcal{E}).$$

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