ζ-integrals and ϵ-functions

The Langlands–Deligne local constant

David Kurniadi Angdinata

University of East Anglia

Tuesday, 26 May 2026

L-functions

Let F be a local field, and let $\chi : F^\times \to \mathbb{C}^\times$ be a quasi-character of F.

- If $F = \mathbb{R}$, then $\chi := \chi_{N,s} := x^{-N}|x|^s$ for some $N \in \{0, 1\}$, and
$$L(\chi) := \Gamma_{\mathbb{R}}(s) := \frac{1}{\pi^{s/2}}\Gamma\left(\frac{s}{2}\right)$$
- If $F = \mathbb{C}$, then $\chi := \chi_{N,s} := z^{-N}|z|^s$ for some $N \in \mathbb{N}$, and
$$L(\chi) := \Gamma_{\mathbb{C}}(s) := \frac{2}{(2\pi)^s}\Gamma(s).$$
- If F is non-archimedean, then
$$L(\chi) := \begin{cases} \dfrac{1}{1 - \chi(\varpi)} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified,} \end{cases}$$

  where ϖ is a uniformiser of F.

ζ-integrals

These L-functions $L(\chi)$ turn out to be special values of ζ-**integrals**

$$\zeta(f,\chi) := \int_{F^\times} f(x)\chi(x)\mathrm{d}^\times x,$$

where f is an appropriate Schwarz–Bruhat **test function** and $\mathrm{d}^\times x$ is a Haar measure on $F^\times$. Since F and $F^\times$ are both locally compact abelian groups, Haar measures exist and are unique up to non-zero scaling.

For simplicity, it suffices to fix convenient choices for each F.

- If $F = \mathbb{R}$, then let $\mathrm{d}x$ be Lebesgue, and let $\mathrm{d}^\times x := \mathrm{d}x/|x|$.
- If $F = \mathbb{C}$, then let $\mathrm{d}z$ be twice Lebesgue, and let $\mathrm{d}^\times z := \mathrm{d}z/|z|$.
- If F is non-archimedean, then choose $\mathrm{d}x$ and $\mathrm{d}^\times x$ such that

  $$\int_{\mathcal{O}} \mathrm{d}x = \int_{\mathcal{O}^\times} \mathrm{d}^\times x = 1,$$

  so that $(1-q^{-1})\mathrm{d}^\times x = \mathrm{d}x/|x|$.

Real ζ-integrals

If $F = \mathbb{R}$, then let $g_N : x \mapsto x^N e^{-\pi x^2}$, so $t = \pi x^2$ gives

$$\begin{aligned}
\zeta(g_N, \chi_{N,s}) &= \int_{\mathbb{R}^\times} x^N e^{-\pi x^2} x^{-N} |x|^s \frac{\mathrm{d}x}{|x|} \\
&= 2 \int_0^\infty e^{-\pi x^2} x^s \frac{\mathrm{d}x}{x} \\
&= 2 \int_0^\infty e^{-t} \left(\frac{t}{\pi}\right)^{s/2} \frac{\mathrm{d}t}{2t} \\
&= \frac{1}{\pi^{s/2}} \Gamma\left(\frac{s}{2}\right) \\
&= L(\chi_{N,s}).
\end{aligned}$$

Complex ζ-integrals

If $F = \mathbb{C}$, then let $h_N : z \mapsto \pi^{-1} z^N e^{-2\pi|z|}$, so $z = re^{i\theta}$ and $t = 2\pi r^2$ give

$$\begin{aligned}
\zeta(h_N, \chi_{N,s}) &= \frac{1}{\pi} \int_{\mathbb{C}^\times} z^N e^{-2\pi|z|} z^{-N} |z|^s \frac{\mathrm{d}z}{|z|} \\
&= \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} e^{-2\pi r^2} r^{2s} \frac{2\mathrm{d}\theta \mathrm{d}r}{r} \\
&= 4 \int_0^\infty e^{-2\pi r^2} r^{2s} \frac{\mathrm{d}r}{r} \\
&= 4 \int_0^\infty e^{-t} \left(\frac{t}{2\pi}\right)^s \frac{\mathrm{d}t}{2t} \\
&= \frac{2}{(2\pi)^s} \Gamma(s) \\
&= L(\chi_{N,s}).
\end{aligned}$$

Similarly, if $\overline{h_N} : z \mapsto \pi^{-1} \overline{z}^N e^{-2\pi|z|}$ and $\overline{\chi_{N,s}} : z \mapsto \overline{z}^{-N} |z|^s$, then

$$\zeta(\overline{h_N}, \overline{\chi_{N,s}}) = L(\overline{\chi_{N,s}}).$$

Unramified non-archimedean ζ-integrals

If F is non-archimedean and χ is unramified, then $x = \varpi^n u$ gives

$$\begin{aligned}
\zeta(\mathbb{I}_{\mathcal{O}}, \chi) &= \int_{F^\times} \mathbb{I}_{\mathcal{O}}(x)\chi(x)\mathrm{d}^\times x \\
&= \int_{\mathcal{O}\setminus\{0\}} \chi(x)\mathrm{d}^\times x \\
&= \sum_{n=0}^{\infty} \int_{\varpi^n\mathcal{O}^\times} \chi(x)\mathrm{d}^\times x \\
&= \sum_{n=0}^{\infty} \chi(\varpi)^n \int_{\mathcal{O}^\times} \chi(u)\mathrm{d}^\times u \\
&= \sum_{n=0}^{\infty} \chi(\varpi)^n \\
&= \frac{1}{1-\chi(\varpi)} \\
&= L(\chi).
\end{aligned}$$

Ramified non-archimedean ζ-integrals

If F is non-archimedean and χ is ramified, then

$$\begin{aligned}\zeta(\mathbb{I}_{1+\varpi^{a(\chi)}\mathcal{O}},\chi) &= \int_{F^\times} \mathbb{I}_{1+\varpi^{a(\chi)}\mathcal{O}}(x)\chi(x)\mathrm{d}^\times x \\ &= \int_{1+\varpi^{a(\chi)}\mathcal{O}} \chi(x)\mathrm{d}^\times x \\ &= \frac{1}{(\mathcal{O}^\times : 1+\varpi^{a(\chi)}\mathcal{O})} \\ &= \frac{1}{(q-1)q^{a(\chi)-1}} \\ &= \frac{1}{q^{a(\chi)}(1-q^{-1})},\end{aligned}$$

where q is the size of the residue field of F.

Fourier transforms

Let f be a Schwartz–Bruhat function on F, let $\psi : F \to \mathbb{C}^\times$ be a non-trivial additive character of F, and let $\mathrm{d}x$ be a Haar measure on F. Then the Fourier transform of f with respect to ψ and $\mathrm{d}x$ is given by

$$\widehat{f}(y) := \int_F f(x)\psi(xy)\mathrm{d}x.$$

If $\widetilde{f}(y)$ is the Fourier transform of f with respect to $\psi \circ c$ and $m\mathrm{d}x$, then

$$\widetilde{f}(y) = \int_F f(x)\psi(cxy)m\mathrm{d}x = m\widehat{f}(cy).$$

For simplicity, it suffices to fix convenient choices for each F.

- If $F = \mathbb{R}$, then let $\psi(x) := e^{2\pi i x}$.
- If $F = \mathbb{C}$, then let $\psi(z) := e^{2\pi i(z+\overline{z})}$.
- If F is non-archimedean, then choose ψ such that $a(\psi) = 0$.

Real Fourier transforms

If $F = \mathbb{R}$, then

$$\begin{aligned}
\widehat{g_N}(y) &= \int_{\mathbb{R}} x^N e^{-\pi x^2} e^{2\pi i x y} \mathrm{d}x \\
&= \frac{1}{(2\pi i)^N} \frac{\mathrm{d}^N}{\mathrm{d}^N y} \int_{\mathbb{R}} e^{-\pi x^2} e^{2\pi i x y} \mathrm{d}x \\
&= \frac{1}{(2\pi i)^N} \frac{\mathrm{d}^N}{\mathrm{d}^N y} e^{-\pi y^2} \\
&= i^N g_N(y).
\end{aligned}$$

Complex Fourier transforms

If $F = \mathbb{C}$, then $z = x + iy$ and $w = u + iv$ give

$$\begin{aligned}
\widehat{h_N}(w) &= \frac{1}{\pi}\int_{\mathbb{C}} z^N e^{-2\pi|z|} e^{2\pi i(zw+\overline{zw})}\mathrm{d}z \\
&= \frac{1}{\pi(2\pi i)^N}\frac{\partial^N}{\partial w^N}\int_{\mathbb{C}} e^{-2\pi|z|} e^{2\pi i(zw+\overline{zw})}\mathrm{d}z \\
&= \frac{1}{\pi(2\pi i)^N}\frac{\partial^N}{\partial w^N}\int_{\mathbb{R}^2} e^{-2\pi(x^2+y^2)} e^{4\pi i(xu-yv)}2\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{\pi(2\pi i)^N}\frac{\partial^N}{\partial w^N}\int_{\mathbb{R}} \sqrt{2}e^{-2\pi x^2} e^{4\pi ixu}\mathrm{d}x \int_{\mathbb{R}} \sqrt{2}e^{-2\pi y^2} e^{4\pi iyv}\mathrm{d}y \\
&= \frac{1}{\pi(2\pi i)^N}\frac{\partial^N}{\partial w^N} e^{-2\pi(u^2+v^2)} \\
&= \frac{i^N}{\pi}\overline{w}^N e^{-2\pi|w|} \\
&= i^N\overline{h_N}(w).
\end{aligned}$$

Non-archimedean Fourier transforms

Assume that F is non-archimedean. If χ is unramified, then

$$\begin{aligned}\widehat{\mathbb{I}_{\mathcal{O}}}(y) &= \int_F \mathbb{I}_{\mathcal{O}}(x)\psi(xy)\,\mathrm{d}x \\ &= \int_{\mathcal{O}} \psi(xy)\,\mathrm{d}x \\ &= \mathbb{I}_{\mathcal{O}}(y).\end{aligned}$$

If χ is ramified, then $x = 1 + u$ gives

$$\begin{aligned}\widehat{\mathbb{I}_{1+\varpi^{a(\chi)}\mathcal{O}}}(y) &= \int_F \mathbb{I}_{1+\varpi^{a(\chi)}\mathcal{O}}(x)\psi(xy)\,\mathrm{d}x \\ &= \int_{1+\varpi^{a(\chi)}\mathcal{O}} \psi(xy)\,\mathrm{d}x \\ &= \psi(y)\int_{\varpi^{a(\chi)}\mathcal{O}} \psi(uy)\,\mathrm{d}u \\ &= \frac{\psi(y)}{q^{a(\chi)}}\mathbb{I}_{\varpi^{-a(\chi)}\mathcal{O}}(y).\end{aligned}$$

Abelian ϵ-functions

It turns out that $\zeta(f,\chi)$ and $L(\chi)$ satisfy the functional equation

$$\frac{\zeta(\widehat{f},\widehat{\chi})}{L(\widehat{\chi})} = \epsilon(\chi,\psi,\mathrm{d}x)\frac{\zeta(f,\chi)}{L(\chi)},$$

where $\widehat{\chi} : x \mapsto |x|/\chi(x)$ is the **twisted dual** of χ.

The **ϵ-function** $\epsilon(\chi,\psi,\mathrm{d}x)$ depends on ψ and $\mathrm{d}x$, and

$$\begin{aligned}\epsilon(\chi,\psi\circ c,\mathrm{d}x) &= \chi(c)|c|^{-1}\epsilon(\chi,\psi,\mathrm{d}x),\\ \epsilon(\chi,\psi,m\mathrm{d}x) &= m\epsilon(\chi,\psi,\mathrm{d}x).\end{aligned}$$

On the other hand, if g is another Schwarz–Bruhat function on F, then

$$\frac{\zeta(f,\chi)}{\zeta(g,\chi)} = \frac{\zeta(\widehat{f},\widehat{\chi})}{\zeta(\widehat{g},\widehat{\chi})},$$

by Fubini's theorem, so $\epsilon(\chi,\psi,\mathrm{d}x)$ is independent of f.

Archimedean and unramified non-archimedean ϵ-functions

If $F = \mathbb{R}$, then $\zeta(g_N, \chi_{N,s}) = L(\chi_{N,s})$ and

$$\zeta(\widehat{g_N}, \widehat{\chi_{N,s}}) = \zeta(i^N g_N, x^N |x|^{1-s}) = i^N \zeta(g_N, \chi_{N,1-s+2N}) = i^N L(\widehat{\chi_{N,s}}),$$

so that $\epsilon(\chi_{N,s}, \psi, \mathrm{d}x) = i^N$.

If $F = \mathbb{C}$, then $\zeta(h_N, \chi_{N,s}) = L(\chi_{N,s})$ and

$$\zeta(\widehat{h_N}, \widehat{\chi_{N,s}}) = \zeta(i^N \overline{h_N}, z^N |z|^{1-s}) = i^N \zeta(\overline{h_N}, \overline{\chi_{N,1-s+N}}) = i^N L(\widehat{\chi_{N,s}}),$$

so that $\epsilon(\chi_{N,s}, \psi, \mathrm{d}z) = i^N$.

If F is non-archimedean and χ is unramified, then $\zeta(\mathbb{I}_{\mathcal{O}}, \chi) = L(\chi)$ and

$$\zeta(\widehat{\mathbb{I}_{\mathcal{O}}}, \widehat{\chi}) = \zeta(\mathbb{I}_{\mathcal{O}}, \widehat{\chi}) = L(\widehat{\chi}),$$

so that $\epsilon(\chi, \psi, \mathrm{d}x) = 1$.

Ramified non-archimedean ϵ-functions

If F is non-archimedean and χ is ramified, then $L(\chi) = L(\widehat{\chi}) = 1$ and

$$\begin{aligned}
\zeta(\widehat{\mathbb{I}_{1+\varpi^{a(\chi)}\mathcal{O}}}, \widehat{\chi}) &= \int_{F^\times} \widehat{\mathbb{I}_{1+\varpi^{a(\chi)}\mathcal{O}}}(y)\widehat{\chi}(y)\mathrm{d}^\times y \\
&= \frac{1}{q^{a(\chi)}} \int_{\varpi^{-a(\chi)}\mathcal{O}\setminus\{0\}} \frac{\psi(y)}{\chi(y)}|y|\mathrm{d}^\times y \\
&= \zeta(\mathbb{I}_{1+\varpi^{a(\chi)}\mathcal{O}}, \chi) \int_{\varpi^{-a(\chi)}\mathcal{O}\setminus\{0\}} \frac{\psi(y)}{\chi(y)}\mathrm{d}y,
\end{aligned}$$

so that

$$\epsilon(\chi, \psi, \mathrm{d}x) = \int_{\varpi^{-a(\chi)}\mathcal{O}\setminus\{0\}} \frac{\psi(y)}{\chi(y)}\mathrm{d}y.$$

In fact,

$$\epsilon(\chi, \psi, \mathrm{d}x) = \int_{\varpi^{-a(\chi)}\mathcal{O}^\times} \frac{\psi(y)}{\chi(y)}\mathrm{d}y.$$

Nonabelian ϵ-functions

Theorem (Langlands–Deligne)

Let F be a local field, let V be a Weil representation of F, let ψ be a non-trivial additive character of F, and let $\mathrm{d}x$ be a Haar measure on F. Then there is a unique constant $\epsilon(V, \psi, \mathrm{d}x) \in \mathbb{C}^\times$ such that

- *if E is a finite separable extension over F and $\mathrm{d}_E x$ is an additive Haar measure on E, then the map*

  $$V \longmapsto \epsilon(V, \psi \circ \mathrm{tr}_F^E, \mathrm{d}_E x)$$

  is inductive in degree zero over F, and
- *if V corresponds to a quasi-character χ of F, then*

  $$\epsilon(V, \psi, \mathrm{d}x) = \epsilon(\chi, \psi, \mathrm{d}x).$$

Uniqueness follows from Brauer's theorem that an inductive function in degree zero is determined by its values on quasi-characters.

Properties of ϵ-functions

Assuming existence, the following are easy consequences.

- $\epsilon(V \oplus W, \psi, \mathrm{d}x) = \epsilon(V, \psi, \mathrm{d}x)\epsilon(W, \psi, \mathrm{d}x)$.
- $\epsilon(V, \psi \circ c, \mathrm{d}x) = (\det V)(c)|c|^{-\dim V}\epsilon(V, \psi, \mathrm{d}x)$.
- $\epsilon(V, \psi, m\mathrm{d}x) = m^{\dim V}\epsilon(V, \psi, \mathrm{d}x)$.
- If F is non-archimedean and W is unramified, then
  $$\epsilon(V \otimes W, \psi, \mathrm{d}x) = \epsilon(V, \psi, \mathrm{d}x)^{\dim W}(\det W)(\varpi)^{a(V)+n(\psi)\dim V}.$$
- If $\widehat{V} := |\cdot|V^*$ is the twisted dual of V, then
  $$\epsilon(V, \psi, \mathrm{d}x)\epsilon(\widehat{V}, \psi \circ -, \mathrm{d}_\psi x) = 1,$$
  where $\mathrm{d}_\psi x$ is the dual Haar measure with respect to ψ.
- If E is a finite separable extension of F and V_E is a degree zero Weil representation of E, then
  $$\epsilon(\mathrm{Ind}_F^E V_E, \psi, \mathrm{d}x) = \epsilon(V_E, \psi \circ \mathrm{tr}_F^E, \mathrm{d}_E x).$$